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Modeling pandemic mortality risk and its application to ortality-linked security pricing

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Highlights

- We propose a mortality model that depicts the relevant pandemic effects using a threshold jump approach.
- We consider mortality jumps related to the pandemic shock and to a specific country shock to capture pandemic mortality dynamics across countries.
- Pandemic jump occurs only when a pandemic event causes significant deaths worldwide.
- We analyze the effect of pandemic mortality risk on pricing a mortality-linked bond using the first Swiss Re mortality bond as an example.

Modeling Pandemic Mortality Risk and its Application to Mortality-Linked Security Pricing

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Modeling Pandemic Mortality Risk and its Application to Mortality-Linked Security Pricing

Abstract

To provide insights for how to deal with pandemic mortality risk, this article introduces a mortality model that depicts the relevant pandemic effects on pricing mortality-linked securities, using a threshold jump approach. That is, to capture pandemic mortality dynamics across countries, we consider mortality jumps related to the pandemic shock and to a specific country shock. Pandemic jump occurs only when a pandemic event causes significant deaths worldwide, such as 1918 Spanish flu or COVID-19. Then the proposed pandemic mortality model can be adjusted according to country-specific mortality experiences. We further analyze the effect of pandemic mortality risk on pricing a mortality-linked bond. Using the first Swiss Re mortality bond as an example, a further derivation obtains the closed-form solution for the fixed-coupon mortality-linked bond in the pandemic mortality framework. Finally, this study details the impacts of pandemic mortality risk numerically by fitting the model to the United States, England and Wales, France, Italy, and Switzerland and calculating the fair spread of the mortality-linked bond.

Keywords: Pandemic Mortality Risk, COVID-19, Wang Transform, Mortality-Linked Security, Threshold Jump Approach

JEL codes: G13, G17, G22

1. Introduction

Mortality uncertainty is the primary source of risk for life insurers and annuity providers. The financial capacity of the life insurance industry to pay catastrophic death losses from hurricanes, epidemics, earthquakes, and other natural or man-made disasters is limited. Securitization of mortality risk is an capital solution for dealing with such risk. As Jaffee and Russell (1997) and Froot (2001) explain, insurance securitization may be a more efficient mechanism for financing catastrophic losses than traditional reinsurance, because it introduces more capital and thereby enhances the ability of insurers to deal with huge losses due to natural disasters like pandemics, hurricanes, and earthquakes. Blake et al. (2008) caution that traditional insurance methods for managing this risk suffer constrained capacity, so capital markets are needed to find effective solutions, as manifested in issuances by the Swiss Reinsurance Company, the world's second-largest reinsurance firm. It issued a three-year mortality-linked bond in 2003 (Vita Capital I), with \$400 million in coverage for institutional investors, then a second bond (Vita Capital II) in 2008. Both these mortality securities aimed to transfer mortality risk away from the insurer, using a combined mortality index that measures annual population mortality in five countries and applies predetermined weights to each nation's publicly reported mortality data. Vita Capital I used the annual population mortality rates for France, United Kingdom, the United States, Italy, and Switzerland; Vita Capital II instead relied on annual population mortality rates for the United States, United Kingdom, Japan, Canada, and Germany. In addition, Swiss Re obtained US\$200 million in coverage against North Atlantic hurricane and UK extreme mortality risk through its Mythen Re program, introduced in 2012. It represents the first combination of hurricane and mortality risks in a bond offering. As these developments indicate, mortality securitization appears to be gaining popularity among life insurers as a tool to transfer huge mortality losses to financial markets.

However, coronavirus Disease 2019 (COVID-19) are emerging as a threat to people in the 21st century. They are responsible for high mortality rates. Huynh et al. (2013) pointed out that a pandemic is an outbreak of infectious disease that spreads throughout the world and infects a significant proportion of the human population. Pandemics arising from influenza are considered the most significant threat to the life insurance industry. Therefore, pricing a mortality-linked bond also may benefit from a greater understanding of pandemic events and their influence on mortality uncertainty.

In the past two decades, various mortality models have been proposed to reflect the dynamics of mortality over time. For example, Lee and Carter (1992) pioneered the modeling of central mortality rates, with log-linear correlations with a time-dependent mortality factor, adjusted for age-specific effects, using two sets of age-dependent coefficients. Cairns et al. (2006) propose a two-factor stochastic mortality model (CBD model) for higher ages and examine the pricing of longevity bonds. The Lee-Carter and CBD models both project mortality rates based on age and period effects. Renshaw and Haberman (2006) extend the Lee-Carter model to consider cohort effects in mortality modeling. Despite their contributions though, these early mortality models exclude pandemic mortality risk and cannot explicitly capture structural changes or pandemic mortality shocks that could cause mortality jumps, such as occurred due to the 1918 worldwide flu pandemic or COVID-19. Research increasingly acknowledges mortality jumps though, including Cox et al. (2006), Lin and Cox (2008), Chen and Cox (2009), Wang et al. (2013), Deng et al. (2012), Zhou et al. (2013), Lin et al. (2013), and Chen (2014). Yang et al. (2010) use principal component analysis (PCA), and Milidonis et al. (2011) employ a Markov regime-switching model to describe the phenomenon of structural changes in mortality rates. Few studies address the potential impacts of mortality shocks across countries though, with some notable exceptions. That is, Zhou et al. (2013) propose a two-population generalization of the model offered by Chen and

Cox (2009) to model transitory mortality jumps. Cox et al. (2006) decompose mortality shocks into a specific factor and a common factor. The common factor appears more substantial, such that it causes the co-movement of the mortality indices in all countries. Lin et al. (2013) also extend Cox et al.'s (2006) model to a general setting and disentangle transient jumps from persistent volatilities. Unlike Cox et al. (2006), who regard unanticipated mortality jumps as permanent shocks, Lin et al. (2013) model them as transient jumps, with a double-jump process. Cox et al. (2006) and Lin et al. (2013) both anticipate that the co-movement of the jump effect is a common factor in all countries. Their models imply that mortality jumps occur simultaneously in all countries.

Another view reflects a potential dependence structure of mortality rates across countries. For example, Li and Hardy (2011) expand the Lee-Carter model to emphasize the importance of mortality modeling for two populations. Yang and Wang (2013) provide multi-country mortality dynamics to capture mortality dependence across countries, and Wang et al. (2015) introduce mortality dependence to multi-country mortality modeling with a dynamic copula approach. Yet in some cases, the movement trend of catastrophic mortality rates across countries might not be modeled properly as a common factor. In this scenario, the mortality shock occurs only if a pandemic event causes significant worldwide deaths. Consider Figure 1, which depicts death trends in France, England and Wales, Italy, Spain, Switzerland, Canada, and the United States, Taiwan from 1816 to 2021. In 2002, SARS killed 775 people in Europe, Asia, and America. The outbreak of the Ebola virus in early 2014 had the most severe effects in Africa; deaths in France, England and Wales, Italy, Switzerland, and the United States revealed no significant jump trend. Conversely, we find clear jump phenomena in France, England and Wales, Italy, Switzerland, and the United States during the 1918 Spanish flu, which killed at least 20 million people. The most recent example, involving COVID-19 and its outbreak in early 2020, has had severe effects around the world. The World

Health Organization declared COVID-19 a global pandemic on March 11, 2020; as of September 30, 2021, more than 739 million people in 202 countries have been affected, with more than 5 million deaths worldwide. As Figure 2 shows, COVID-19 causes severe jump of deaths around the world.

As both the 1918 Spanish flu and COVID-19 demonstrate, pandemic mortality risk leads to numerous deaths, and the fatal infectious disease can spread across countries, especially as globalization and improved transportation capabilities encourage its spread. Such patterns indicate that mortality rates across countries should experience significant jump when a mortality shock leads to substantially higher deaths. Accordingly, we refer to this effect as pandemic mortality shock. Although pandemic mortality risk clearly exists in reality, and humanity simply cannot afford to ignore it or its impacts for pricing mortality securities, the challenge of modeling this risk has not been addressed thus far.

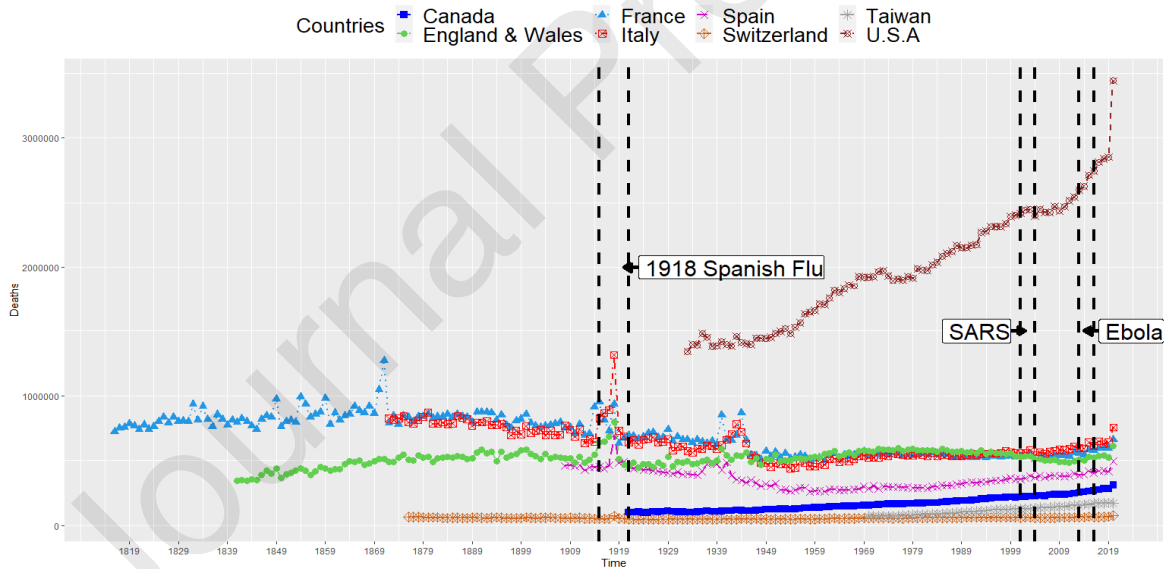


Figure 1: Annual Death Trends in Different Countries, 1816–2019

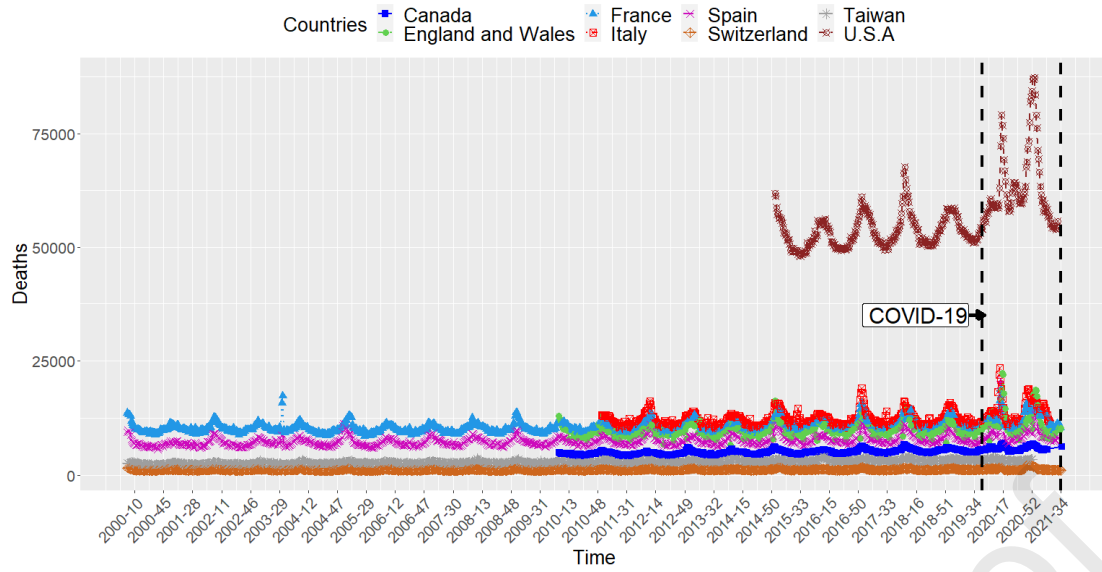


Figure 2: Weekly Death Trends in Different Countries, 2000–2021

In response, we seek to perform pandemic mortality modeling and determine the effects of pandemic mortality risk on pricing a mortality-linked security. In so doing, we establish three main contributions. First, we present an initial, multi-country pandemic mortality model that captures jump phenomena during a pandemic event. We consider a specific country mortality shock and a pandemic mortality shock. The pandemic mortality shock occurs only if a pandemic event causes significant worldwide deaths and the new death numbers in the country are greater or equal to the average new death in the world. On the contrary, if the new death numbers in the country are not greater or equal to the average new death in the world, the pandemic jump will not happen in that country. Second, using the first Swiss Re mortality bond as an example, we obtain a closed-form solution with a Wang (2000) transform, in line with Cox et al. (2006) and Denuit et al. (2007). Thus, we can establish the effect of pandemic mortality risk on a mortality-linked bond. Third, with an empirical study, we compare the impacts of pandemic mortality risk on a mortality-linked security pricing in the full period, pre-COVID-19 period, and during COVID-19 period. The findings provide novel insights into mortality-linked security pricing in a post-pandemic world.

The remainder of this paper is organized as follows: In Section 2, we propose a pandemic mortality model to capture the effect of the pandemic mortality rate across countries. Section 3 presents an analytical solution to the mortality-linked security, Swiss Re bond through the Wang (2000) transform, and Denuit et al. (2007) approach. After adopting a calibration approach to estimate model variables, we offer empirical and sensitivity analyses in Section 4. Section 5 contains the conclusions.

2. Modeling Pandemic Mortality

To capture the pandemic mortality risk, we propose a mortality model that considers a specific country mortality shock and a pandemic mortality shock governing the mortality dynamics first. We define the notations for modeling these two types of shocks first.

- (1) $q_{i,t}$: the mortality rate at time t for the i^{th} country;
- (2) $N_{i,t}$: the new deaths of the i^{th} country from a pandemic event at time t ;
- (3) N_t : the average new deaths of the world from a pandemic event at time t ;
- (4) I_t^i : the jump frequency of the i^{th} country at time t caused by $N_{i,t} \geq N_t$, which captures the pandemic jump; a pandemic mortality jump occurs only if a pandemic event causes significant worldwide deaths.
- (5) $\Gamma_{i,t}$: the jump frequency resulting from the mortality shock in the i^{th} country at time t , which captures a specific country jump.

Throughout this article, we use (Ω, \mathcal{F}, P) to denote a filtered probability space that accommodates all sources of randomness. We introduce the pricing measure Q first. Let $E^Q[\cdot]$ indicate the conditional expectation under Q , given the information available before time 0. By definition, the original or actual probabilities of an event

represent the measures of their likely occurrence in the real world. Shreve (2004) uses a coin toss as an example of the original probability measure P . Risk-neutral probability differs from the actual probability, in that it removes any trend component from the security, apart from the one given to it by the risk-free rate of growth. As Shreve (2004) explains, in the risk-neutral measure, every asset has a mean return to the interest rate, and the realized risk-neutral return for assets is characterized solely by their volatility vector processes.¹ Thus, the process is a martingale under Q , in which the value of the i^{th} asset is denominated in shares of the money account. Shreve (2004) also denotes the measure Q as risk neutral for the money market account numeraire. Extending the same concept, the multi-country mortality dynamics ($q_{i,t}$) at time t for the i^{th} country considering m countries can be modeled as

$$\begin{aligned} \frac{dq_{1,t}}{q_{1,t}} &= \mu_1 dt + \sigma_1 dW_{1,t} + (\Lambda_1 - 1)d\Gamma_{1,t} + (\pi_1 - 1)dI_t^{\tilde{1}}, \\ \frac{dq_{2,t}}{q_{2,t}} &= \mu_2 dt + \sigma_2 dW_{2,t} + (\Lambda_2 - 1)d\Gamma_{2,t} + (\pi_2 - 1)dI_t^{\tilde{2}}, \text{ and} \\ &\dots\dots\dots \\ \frac{dq_{m,t}}{q_{m,t}} &= \mu_m dt + \sigma_m dW_{m,t} + (\Lambda_m - 1)d\Gamma_{m,t} + (\pi_m - 1)dI_t^{\tilde{m}}, \end{aligned} \quad (1)$$

where μ_i and σ_i are constants, and $W_{i,t}$ is a one-dimensional standard Brownian motion under the original probability measure P at time t . The correlation coefficient between $W_{i,t}$ and $W_{v,t}$ is denoted as $corr(dW_{v,t}, dW_{i,t}) = \rho_{v,i}$ for $v \neq i$, where

¹ See Shreve (2004, p. 377). Alternatively, the forward-looking risk-neutral measure PT is a martingale, using a zero-coupon bond maturing at time T as a numeraire. We can deal with derivative pricing in the PT forward-looking risk-neutral measure if interest rates are stochastic. For the current study, we focus on the impacts of pandemic mortality risk on the fair spread of a fixed-coupon bond, such that we do not consider stochastic interest rates. Accordingly, we employ the risk-neutral measure Q to price the fixed-coupon mortality bond.

i or $v = 1, 2, 3, \dots, m$. The sequence of $\{I_t^{\tilde{i}}\}$ mutually independently follow Poisson distributions with intensities of $\lambda_{\tilde{i}}$ that are nonnegative constants, and $\{\Gamma_{i,t}\}$ also independently takes a Poisson distribution with intensity of λ_{Γ_i} , which we assume to be a nonnegative constant. Furthermore, $\{I_t^{\tilde{i}}\}$ is mutually independent of $\{\Gamma_{i,t}\}$ in $i = 1, 2, 3, \dots, m$. Both $I_t^{\tilde{i}}$ and $\Gamma_{i,t}$ are independent Poisson-jump processes, driven, respectively, by the pandemic mortality risk and the specific country risk at time t . We calculate the pandemic new deaths of the i^{th} country and the average new deaths of the world in one week in the empirical analysis.

Following traditional predictions about jumps by Merton (1976), we present $\pi_i - 1$ as the random variable percentage in the mortality rate of the i^{th} country that results from pandemic jumps of deaths in the world. The natural logarithm of π_i , or the jump amplitude driven by deaths in other countries, should follow a normal distribution with a mean of u_{π_i} and variance of $\sigma_{\pi_i}^2$, which also can be denoted $\ln \pi_i \sim N(u_{\pi_i}, \sigma_{\pi_i}^2)$, $\pi_i > 0$, $i = 1, 2, 3, \dots, m$. In contrast, $\Lambda_i - 1$ refers to the percentage of the mortality rate of the i^{th} country that results from specific jumps in deaths of the i^{th} country. The natural logarithm of specific jump size is distributed as a normal random variable, $\ln \Lambda_i \sim N(u_{\Lambda_i}, \sigma_{\Lambda_i}^2)$, $\Lambda_i > 0$, $i = 1, 2, 3, \dots, m$. Again using Merton's (1976) traditional assumptions, we anticipate that $W_{i,t}$, Λ_i , π_i , $I_t^{\tilde{i}}$, and $\Gamma_{i,t}$ are pairwise independent in $i = 1, 2, 3, \dots, m$ at time t . Finally, let $v_{\Lambda_i} \equiv E(\Lambda_i - 1) = \exp(u_{\Lambda_i} + \frac{1}{2}\sigma_{\Lambda_i}^2) - 1$, and $v_{\pi_i} \equiv E(\pi_i - 1) = \exp(u_{\pi_i} + \frac{1}{2}\sigma_{\pi_i}^2) - 1$, $i = 1, 2, 3, \dots, m$.

From Equation (1), we know that $\ln \pi_i$ signifies the impact magnitude of the pandemic jumps of the i^{th} country driven by deaths from a pandemic event. When $d\tilde{I}_t^i$ is 0, there is no pandemic effect on mortality rates for $i = 1, 2, 3, \dots, m$ in Equation (1).

3. Valuation of a Mortality-Linked Bond in the Presence of Pandemic Mortality Risk

3.1 Structure of a Mortality-Linked Bond

We analyze the effect of pandemic mortality risk on pricing a mortality-linked bond. Mortality bonds transfer mortality risk to investors in the capital market. We use the first Swiss Re mortality bond as an example, which is issued in late December 2003 and matured on January 1, 2007 (Swiss Re, 2003). It is a three-year bond. This bond pays a fixed annual coupon (C) and the principal (F) is exposed to mortality risk, linked to the mortality indices. The Swiss Re is based on the average annual population mortality rates in the United States, England and Wales, France, Italy, and Switzerland. If this rate exceeded 130% of the actual 2002 level, investors received a reduced principal payment at maturity. We let B_T denote the principal payment at maturity time T , expressed as

$$B_T = \text{Max}(1 - \text{Loss}, 0), \quad (2)$$

$$\text{with } \text{Loss} = \frac{\text{Max}(Y_{\max} - 1.3Y_{t_0}, 0) - \text{Max}(Y_{\max} - 1.5Y_{t_0}, 0)}{0.2Y_{t_0}}, \text{ and}$$

$$Y_{\max} = \text{Max}(Y_{t_1}, Y_{t_2}, Y_{t_3}) \text{ and } Y_{t_i} = (q_{1,t_i}^{a_1} q_{2,t_i}^{a_2} \dots q_{5,t_i}^{a_5})^{\frac{1}{a_1 + a_2 + \dots + a_5}},$$

According to the design of Swiss Re mortality bond (Swiss Re, 2003), $Y_{t_0}, Y_{t_1}, Y_{t_2}, Y_{t_3}$,

Y_{t_2} , and Y_{t_3} stand for the geometric average population mortality index of the focal countries in 2002, 2003, 2004, 2005, and 2006, respectively. Furthermore, $q_{1,t}$, $q_{2,t}$, ..., and $q_{5,t}$ represent the mortality rates for five counties of the United States, England and Wales, France, Italy, and Switzerland, respectively, and $a_1, a_2, ..., a_4$ and a_5 indicate the weights of their population mortality indices, also respectively.

The present value of the expected cash flow of the fixed-coupon mortality bonds for investors is

$$\begin{aligned} B_0 &= F \times e^{-r(t_3-t_0)} E^Q[B_T] + C \left[e^{-r(t_1-t_0)} + e^{-r(t_2-t_0)} + e^{-r(t_3-t_0)} \right] \\ &= F \times e^{-r(t_3-t_0)} E^Q \left[\text{Max} \left(1 - \frac{\text{Max}(Y_{\max} - K_1, 0) - \text{Max}(Y_{\max} - K_2, 0)}{K_2 - K_1}, 0 \right) \right] \\ &\quad + C \left[e^{-r(t_1-t_0)} + e^{-r(t_2-t_0)} + e^{-r(t_3-t_0)} \right], \end{aligned} \quad (3)$$

where $E^Q(\cdot)$ denotes the expectation value under the risk-neutral probability measure Q at time t_0 , r is the constant risk-free rate, and $K_1 = 1.3 Y_{t_0}$ and $K_2 = 1.5 Y_{t_0}$, with $K_2 > K_1$. We provide a general valuation formula for a mortality bond with K_1 and K_2 , which can be structured to reflect different payoffs for the mortality bond. However, investors must pay the face value if the mortality bonds are issued at par. In turn, $B_0 = F$, and we can obtain the fair spread (C^*), which also can be denoted as

$$C^* = \frac{F - F \times e^{-r(t_3-t_0)} E^Q \left[\text{Max} \left(1 - \frac{\text{Max}(Y_{\max} - K_1, 0) - \text{Max}(Y_{\max} - K_2, 0)}{K_2 - K_1}, 0 \right) \right]}{e^{-r(t_1-t_0)} + e^{-r(t_2-t_0)} + e^{-r(t_3-t_0)}}. \quad (4)$$

3.2 Valuation Formula for a Mortality-Linked Bond

Pricing derivative securities in the complete market involves replicating portfolios. If there is a traded bond and stock index, options on the stock index can be replicated

by holding bonds and the index, which are priced. The mortality bond is a mortality derivative, but there is no efficiently traded mortality index with which to create a replicating hedge. To deal with pricing in such an incomplete market, the Wang (2000) transform offers a popular option that relies on the following: For a risk with a cumulative density function (CDF) $F(x)$ under the original probability measure P , the risk-adjusted CDF $F^*(x)$ under the risk-neutral probability measure Q for pricing risk is given by

$$F^*(x) = \Phi(\Phi^{-1}(F(x)) + \theta), \quad (5)$$

where θ is a constant risk premium, and $\Phi(\cdot)$ is a cumulative standard normal probability.

Furthermore, Denuit et al. (2007) extend the Wang (2000) transform and show that the expected value of a random variable X_t under the risk-neutral probability measure Q can be given by

$$E^Q[X_t] = \int_0^1 [1 - \Phi(\Phi^{-1}(F(p)) + \theta)] dp, \quad (6)$$

in which $0 \leq p \leq 1$, and θ is a risk premium.

We apply Equations (5) and (6) to solve Equation (4) for pricing a fixed-coupon mortality bond (i.e., Swiss Re bond). We denote the total risks at time t in the i^{th} country as $X_{i,t} = \Gamma_{i,t} + I_t^i$, which follows a Poisson-jump process with intensity $\lambda_{X_{i,t}}$. Suppose $x_i - 1$ is the percentage of the mortality rate of the i^{th} country resulting from total risks, and x_i follows normal distributions with a mean of u_{x_i} and variance of $\sigma_{x_i}^2$, and $v_{x_i} \equiv E[(x_i - 1)] = \exp(u_{x_i} + \frac{1}{2}\sigma_{x_i}^2) - 1$. In addition,

$$(x_i - 1)dX_{i,t} \stackrel{d}{=} (\Lambda_i - 1)d\Gamma_{i,t} + (\pi_i - 1)dI_t^i. \quad (7)$$

Thus, we obtain

$$E[(x_i - 1)dX_{i,t}] = E[(\Lambda_i - 1)d\Gamma_{i,t} + (\pi_i - 1)dI_t^i], \text{ and (8)}$$

$$Var[(x_i - 1)dX_{i,t}] = Var[(\Lambda_i - 1)d\Gamma_{i,t} + (\pi_i - 1)dI_t^i]. \text{ (9)}$$

Using Equations (8) and (9), we then obtain:

$$E[x_i - 1] = \frac{(e^{\frac{u_{\pi_i}}{2} + \frac{1}{2}\sigma_{\pi_i}^2} - 1)\lambda_{I^i} + (e^{\frac{u_{\Lambda_i}}{2} + \frac{1}{2}\sigma_{\Lambda_i}^2} - 1)\lambda_{\Gamma_i}}{\lambda_{\Gamma_i} + \lambda_{I^i}},$$

$$\Rightarrow u_{x_i} = \ln \left(\frac{(e^{\frac{u_{\pi_i}}{2} + \frac{1}{2}\sigma_{\pi_i}^2} - 1)\lambda_{I^i} + (e^{\frac{u_{\Lambda_i}}{2} + \frac{1}{2}\sigma_{\Lambda_i}^2} - 1)\lambda_{\Gamma_i}}{\lambda_{\Gamma_i} + \lambda_{I^i}} + 1 \right) - \frac{1}{2}\sigma_{x_i}^2, \text{ (10)}$$

$$\sigma_{x_i}^2 = \ln \left(\frac{B}{\lambda_{\Gamma_i} + \lambda_{I^i}} + \left(\frac{(e^{\frac{u_{\pi_i}}{2} + \frac{1}{2}\sigma_{\pi_i}^2} - 1)\lambda_{I^i} + (e^{\frac{u_{\Lambda_i}}{2} + \frac{1}{2}\sigma_{\Lambda_i}^2} - 1)\lambda_{\Gamma_i}}{\lambda_{\Gamma_i} + \lambda_{I^i}} + 1 \right)^2 \right) - \left(\frac{(e^{\frac{u_{\pi_i}}{2} + \frac{1}{2}\sigma_{\pi_i}^2} - 1)\lambda_{I^i} + (e^{\frac{u_{\Lambda_i}}{2} + \frac{1}{2}\sigma_{\Lambda_i}^2} - 1)\lambda_{\Gamma_i}}{\lambda_{\Gamma_i} + \lambda_{I^i}} + 1 \right)^2$$

$$- 2 \ln \left(\frac{(e^{\frac{u_{\pi_i}}{2} + \frac{1}{2}\sigma_{\pi_i}^2} - 1)\lambda_{I^i} + (e^{\frac{u_{\Lambda_i}}{2} + \frac{1}{2}\sigma_{\Lambda_i}^2} - 1)\lambda_{\Gamma_i}}{\lambda_{\Gamma_i} + \lambda_{I^i}} + 1 \right), \text{ (11)}$$

with

$$B = \left[e^{2u_{\Lambda_i} + 2\sigma_{\Lambda_i}^2} - e^{2u_{\Lambda_i} + \sigma_{\Lambda_i}^2} + \left(e^{\frac{u_{\Lambda_i}}{2} + \frac{1}{2}\sigma_{\Lambda_i}^2} - 1 \right)^2 \right] \lambda_{\Gamma_i}$$

$$+ \left[e^{2u_{\pi_i} + 2\sigma_{\pi_i}^2} - e^{2u_{\pi_i} + \sigma_{\pi_i}^2} + \left(e^{\frac{u_{\pi_i}}{2} + \frac{1}{2}\sigma_{\pi_i}^2} - 1 \right)^2 \right] \lambda_{I^i}.$$

Under the original probability measure P , by Ito's lemma, Equation (1) can be

rewritten as $q_{i,T} = q_{i,t_0} e^{(\mu_i - \frac{1}{2}\sigma_i^2)(T-t_0) + \sigma_i W_{i,T-t_0}} \prod_{l=1}^{X_{i,T}} x_{i,l}$, $i = 1, 2, 3, \dots, m$, in which $x_{i,l}$

represents the jump size, driven by both specific risks and pandemic risks for each

country, and $X_{i,T}$ is the jump number of both risks of each country at time T . Let $\Gamma_{i,t_0} = 0$

and $I_{t_0}^i = 0$, such that $X_{i,t_0} = 0$, because $X_{i,t_0} = \Gamma_{i,t_0} + I_{t_0}^i$.

The form $q_{i,T} = q_{i,t_0} e^{(\mu_i - \frac{1}{2}\sigma_i^2)(T-t_0) + \sigma_i W_{i,T-t_0}} \prod_{l=1}^{X_{i,T}} x_{i,l}$ follows the jump model presented by

Merton (1976) and Kou (2002). The i stands for the United States, England and Wales, France, Italy, or Switzerland, respectively. Also,

$$\ln q_{i,T} = \ln q_{i,t_0} + (\mu_i - \frac{1}{2}\sigma_i^2)(T-t_0) + \sigma_i W_{i,T-t_0} + \sum_{l=1}^{X_{i,T}} \ln x_{i,l}. \quad (12)$$

Next, let X_t represent the sum of the pandemic mortality risk and the specific country jump risks for the United States, England and Wales, France, Italy, and Switzerland, such that $X_t = X_{1,t} + X_{2,t} + \dots + X_{5,t}$, which follows a Poisson distribution with intensity λ_{X_t} and $\lambda_{X_t} = \sum_{i=1}^5 (\lambda_{i^i} + \lambda_{i^i})$. For simplicity, we assume the sequences of $\{X_{i,t}\}$ and $\{W_{i,t}\}$ are mutually independent. To derive the closed-form solution of the fair spread of the mortality bond, we offer Proposition 1.

Proposition 1. Let Z_t be a random variable describing the jump size driven by total pandemic jumps and specific country jumps for the United States, England and Wales, France, Italy, and Switzerland, where $\ln Z$ follows a normal distribution with a mean of μ_z and variance of σ_z^2 . Given the respective logarithm of the mortality indices for the United States, England and Wales, France, Italy, and Switzerland, as in Equation (13), the logarithm of the geometric average population mortality rates of the five countries takes the following form:

$$\ln Y_T = \ln Y_{t_0} + \mu_y(T-t_0) + \sigma_y W_{T-t_0}^Q + a \sum_{l=1}^{X_T} \ln Z_l, \quad (13)$$

where $W_{T-t_0}^Q$ is a one-dimensional standard Brownian motion defined in a filtered

probability space (Ω, F, Q) under the risk-neutral probability, Q ;

$$a = \frac{1}{a_1 + a_2 + \dots + a_5};$$

$$\mu_y = a \left[a_1 \left(\mu_1 - \frac{1}{2} \sigma_1^2 - \sigma_1 \theta_w (1 + \rho_{2,1} + \rho_{3,1} + \rho_{4,1} + \rho_{5,1}) \right) + a_2 \left(\mu_2 - \frac{1}{2} \sigma_2^2 - \sigma_2 \theta_w (1 + \rho_{1,2} + \rho_{3,2} + \rho_{4,2} + \rho_{5,2}) \right) + \dots + a_5 \left(\mu_5 - \frac{1}{2} \sigma_5^2 - \sigma_5 \theta_w (1 + \rho_{1,5} + \rho_{2,5} + \rho_{3,5} + \rho_{4,5}) \right) \right];$$

$$\sigma_y = a \sqrt{[a_1 \sigma_1 \ a_2 \sigma_2 \ a_3 \sigma_3 \ a_4 \sigma_4 \ a_5 \sigma_5] \begin{pmatrix} 1 & \dots & \rho_{15} \\ \vdots & \ddots & \vdots \\ \rho_{51} & \dots & 1 \end{pmatrix} [a_1 \sigma_1 \ a_2 \sigma_2 \ a_3 \sigma_3 \ a_4 \sigma_4 \ a_5 \sigma_5]^T};$$

$$\text{and } \sum_{l=1}^{X_T} \ln Z_l = a_1 \sum_{l=1}^{X_{1,T}} \ln x_{1,l} + a_2 \sum_{l=1}^{X_{2,T}} \ln x_{2,l} + \dots + a_5 \sum_{l=1}^{X_{5,T}} \ln x_{5,l}.$$

Proof. See Appendix A.

From Proposition 1, we know that if X_{t_j} is any constant ($X_{t_j} = k_j$),

$\ln Z_{t_j} | X_{t_j} = k_j$ has a normal distribution with mean u_z and variance σ_z^2 in $j = 1, 2, 3$.

When $X_t = X_{1,t} + X_{2,t} + \dots + X_{5,t}$, and $X_{i,t} = s_i$, such that s_i is any constant $i = 1, 2, \dots, 5$,

we obtain $u_z = \frac{\sum_{i=1}^5 s_i a_i u_{x_i}}{k}$ and $\sigma_z^2 = \frac{\sum_{i=1}^5 s_i a_i^2 \sigma_{x_i}^2}{k}$ with $X_t = k$. For Equation (4), we

suppose $S_T = \frac{\text{Max}(Y_{\max} - K_1, 0) - \text{Max}(Y_{\max} - K_2, 0)}{K_2 - K_1}$.

Conditional on $Y_{\max} = Y_{t_j}$, we obtain

$$S_T = \frac{(Y_{t_j} - K_1)1_{\{Y_{t_j} > K_1\}} - (Y_{t_j} - K_2)1_{\{Y_{t_j} > K_2\}}}{K_2 - K_1}, \text{ for } j = 1, 2, 3. \quad (14)$$

If $Y_{max} = Y_{t_j}$, then $S_T = \frac{(Y_{t_j} - K_1)1_{\{Y_{t_j} > K_1\}} - (Y_{t_j} - K_2)1_{\{Y_{t_j} > K_2\}}}{K_2 - K_1}$ in $j = 1, 2, 3$. Therefore,

Equation (4) can be rewritten as

$$\begin{aligned} C^* &= \frac{F - F \times e^{-r(t_3-t_0)} E^Q \left[(1 - S_T) 1_{\{S_T < 1\}} \right]}{e^{-r(t_1-t_0)} + e^{-r(t_2-t_0)} + e^{-r(t_3-t_0)}} \\ &= \frac{F - F \times e^{-r(t_3-t_0)} \left[E^Q \left(1_{\{S_T < 1\}} \right) - E^Q \left(S_T 1_{\{S_T < 1\}} \right) \right]}{e^{-r(t_1-t_0)} + e^{-r(t_2-t_0)} + e^{-r(t_3-t_0)}}, \end{aligned} \quad (15)$$

where

$$\begin{aligned} &E^Q \left(1_{\{S_T < 1\}} \right) \\ &= E^Q \left[1_{S_T < 1} \middle| Y_{max} = Y_{t_1} \right] * P_r^Q(Y_{max} = Y_{t_1}) + E^Q \left[1_{S_T < 1} \middle| Y_{max} = Y_{t_2} \right] * P_r^Q(Y_{max} = Y_{t_2}) \\ &\quad + E^Q \left[1_{S_T < 1} \middle| Y_{max} = Y_{t_3} \right] * P_r^Q(Y_{max} = Y_{t_3}). \end{aligned} \quad (16)$$

From Equation (15), to obtain the closed-form solution with Equations (5) and (6), we must solve the probability that $Y_{max} = Y_{t_j}$ under the risk-neutral probability measure.

Therefore, Proposition 2 is necessary.

Proposition 2. Given $Y_{max} = \text{Max}(Y_{t_1}, Y_{t_2}, Y_{t_3})$ and $X_{t_j} = k_j$, the probability that

$Y_{max} = Y_{t_j}$ for $j = 1, 2, 3$ under the risk-neutral measure Q is:

$$\begin{aligned} &P_r^Q(Y_{max} = Y_{t_1}) \\ &= \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y(t_2-t_1) - a(k_2-k_1)u_z}{\sqrt{\sigma_y^2(t_2-t_1) + a^2(k_2-k_1)\sigma_z^2}} \right) \right] \\ &\quad \times \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3-t_2) - a(k_3-k_2)u_z}{\sqrt{\sigma_y^2(t_3-t_2) + a^2(k_3-k_2)\sigma_z^2}} \right) \right] \\ &\quad + \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3-t_1) - a(k_3-k_1)u_z}{\sqrt{\sigma_y^2(t_3-t_1) + a^2(k_3-k_1)\sigma_z^2}} \right) \right] \\ &\quad \times \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\mu_y(t_3-t_2) + a(k_3-k_2)u_z}{\sqrt{\sigma_y^2(t_3-t_2) + a^2(k_3-k_2)\sigma_z^2}} \right) \right], \end{aligned} \quad (17)$$

$$\begin{aligned}
P_r^Q(Y_{max} = Y_{t_2}) = & \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \Phi \left(\frac{\mu_y(t_2 - t_1) + a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right] \\
& \times \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3 - t_1) - a(t_3 - t_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right] \\
& + \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3 - t_2) - a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right] \\
& \times \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\mu_y(t_3 - t_1) + a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right], \tag{18}
\end{aligned}$$

and

$$\begin{aligned}
P_r^Q(Y_{max} = Y_{t_3}) = & \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\mu_y(t_3 - t_2) + a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right] \\
& \times \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\mu_y(t_2 - t_1) + a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right] \\
& + \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\mu_y(t_3 - t_1) + a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right] \\
& \times \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right]. \tag{19}
\end{aligned}$$

Proof. See Appendix B.

Let $Y_{max} = Y_{t_j}$ for $j = 1, 2, 3$. Then we can obtain Equation (20):

$$\begin{aligned}
& E^Q \left[1_{S_r < 1} \middle| Y_{\max} = Y_{t_j} \right] \\
&= P_r^Q \left(\frac{(Y_{t_j} - K_1) 1_{\{Y_{t_j} > K_1\}} - (Y_{t_j} - K_2) 1_{\{Y_{t_j} > K_2\}}}{K_2 - K_1} < 1 \middle| Y_{\max} = Y_{t_j} \right) \\
&= \frac{P_r^Q(Y_{\max} = Y_{t_j}) - P_r^Q(Y_{\max}, Y_{t_j} > K_2)}{P_r^Q(Y_{\max} = Y_{t_j})} \quad (20) \\
&= 1 - \frac{P_r^Q(Y_{\max} = Y_{t_j}, Y_{t_j} > K_2)}{P_r^Q(Y_{\max} = Y_{t_j})} \text{ for } j = 1, 2, 3.
\end{aligned}$$

Suppose that $Y_{\max} = Y_{t_1}$, Y_{t_2} , or Y_{t_3} . Then Equations (21)-(23) can be obtained.

$$\begin{aligned}
& P_r^Q(Y_{\max} = Y_{t_1}, Y_{t_1} > K_2) \\
&= P_r^Q(Y_{t_1} > Y_{t_3}, Y_{t_1} > K_2) \left[1 - P_r^Q(Y_{t_3} \leq Y_{t_2}) \right] + P_r^Q(Y_{t_1} > Y_{t_2}, Y_{t_1} > K_2) P_r^Q(Y_{t_3} \leq Y_{t_2}), \quad (21)
\end{aligned}$$

$$\begin{aligned}
& P_r^Q(Y_{\max} = Y_{t_2}, Y_{t_2} > K_2) \\
&= P_r^Q(Y_{t_2} > Y_{t_3}, Y_{t_2} > K_2) \left[1 - P_r^Q(Y_{t_3} \leq Y_{t_1}) \right] + P_r^Q(Y_{t_2} > Y_{t_1}, Y_{t_2} > K_2) P_r^Q(Y_{t_1} \geq Y_{t_3}), \quad (22)
\end{aligned}$$

and

$$\begin{aligned}
& P_r^Q(Y_{\max} = Y_{t_3}, Y_{t_3} > K_2) \\
&= P_r^Q(Y_{t_3} > Y_{t_2}, Y_{t_3} > K_2) \left[1 - P_r^Q(Y_{t_2} \leq Y_{t_1}) \right] + P_r^Q(Y_{t_3} > Y_{t_1}, Y_{t_3} > K_2) P_r^Q(Y_{t_1} \geq Y_{t_2}). \quad (23)
\end{aligned}$$

Further, $P_r^Q(Y_{\max} = Y_{t_1}, Y_{t_1} > K_2)$, $P_r^Q(Y_{\max} = Y_{t_2}, Y_{t_2} > K_2)$ and

$P_r^Q(Y_{\max} = Y_{t_3}, Y_{t_3} > K_2)$ can be respectively derived in Equations (24)-(26) as shown

in Appendix C in detail.

$$\begin{aligned}
P_r^Q(Y_{max} = Y_{t_1}, Y_{t_1} > K_2) = & \\
& \left\{ \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3-t_1) - a(k_3-k_1)u_z}{\sqrt{\sigma_y^2(t_3-t_1) + a^2(k_3-k_1)\sigma_z^2}} \right) \right. \\
& \left. - \sum_{k_3=0}^{\infty} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\ln K_2 - \mu_y(t_3-t_0) - \ln Y_{t_0} - a k_3 u_z}{\sqrt{\sigma_y^2(t_3-t_0) + a^2 k_3 \sigma_z^2}} \right) \right. \\
& \left. + \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi(-d_1, d_3, \rho_{1,3}^y) \right\} \\
& \times \left[1 - \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3-t_2) - a(k_3-k_2)u_z}{\sqrt{\sigma_y^2(t_3-t_2) + a^2(k_3-k_2)\sigma_z^2}} \right) \right] \\
& \left\{ \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y(t_2-t_1) - a(k_2-k_1)u_z}{\sqrt{\sigma_y^2(t_2-t_1) + a^2(k_2-k_1)\sigma_z^2}} \right) \right. \\
& \left. - \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\ln K_2 - \mu_y(t_2-t_0) - \ln Y_{t_0} - a k_2 u_z}{\sqrt{\sigma_y^2(t_2-t_0) + a^2 k_2 \sigma_z^2}} \right) \right\} \\
& + \left\{ \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi(-d_1, d_2, \rho_{1,2}^y) \right\} \\
& \times \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3-t_2) - a(k_3-k_2)u_z}{\sqrt{\sigma_y^2(t_3-t_2) + a^2(k_3-k_2)\sigma_z^2}} \right) \right], \quad (24)
\end{aligned}$$

$$\begin{aligned}
P_r^Q(Y_{max} = Y_{t_2}, Y_{t_2} > K_2) = & \\
& \left\{ \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3 - t_2) - a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right. \\
& \left. - \sum_{k_3=0}^{\infty} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\ln K_2 - \mu_y(t_3 - t_0) - \ln Y_{t_0} - a k_3 u_z}{\sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}} \right) \right. \\
& \left. + \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi(-d_2, d_3, \rho_{2,3}^y) \right\} \\
& \times \left[1 - \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right] \\
& \left\{ \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\mu_y(t_2 - t_1) + a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right. \\
& \left. - \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \Phi \left(\frac{\ln K_2 - \mu_y(t_1 - t_0) - \ln Y_{t_0} - a k_1 u_z}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2}} \right) \right. \\
& \left. + \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi(d_1, -d_2, \rho_{1,2}^y) \right\} \\
& \times \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right], \quad (25)
\end{aligned}$$

and

$$\begin{aligned}
P_r^Q(Y_{max} = Y_{t_3}, Y_{t_3} > K_2) = & \left\{ \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\mu_y(t_3 - t_2) + a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right. \\
& \left. - \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\ln K_2 - \mu_y(t_2 - t_0) - \ln Y_{t_0} - a k_2 u_z}{\sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2}} \right) \right. \\
& \left. + \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi(d_2, -d_3, \rho_{2,3}^y) \right\} \\
& \times \left[1 - \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right] \\
& \left\{ \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\mu_y(t_3 - t_1) + a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right. \\
& \left. - \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \Phi \left(\frac{\ln K_2 - \mu_y(t_1 - t_0) - \ln Y_{t_0} - a k_1 u_z}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2}} \right) \right. \\
& \left. + \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi(d_1, -d_3, \rho_{1,3}^y) \right\} \\
& \times \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right], \quad (26)
\end{aligned}$$

in which $d_1 = \frac{\ln K_2 - \mu_y(t_1 - t_0) - \ln Y_{t_0} - a k_1 u_z}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2}},$

$$d_2 = \frac{\ln K_2 - \mu_y(t_2 - t_0) - \ln Y_{t_0} - a k_2 u_z}{\sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2}}, \quad d_3 = \frac{\ln K_2 - \mu_y(t_3 - t_0) - \ln Y_{t_0} - a k_3 u_z}{\sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}},$$

$$u_z = \frac{\sum_{i=1}^5 s_i a_i u_{x_i}}{k}, \quad \sigma_z^2 = \frac{\sum_{i=1}^5 s_i a_i^2 \sigma_{x_i}^2}{k}, \quad \rho_{1,2}^y = \frac{\sigma_y^2(t_1 - t_0) + a^2 k_1 k_2 \sigma_z^2}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2} \sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2}},$$

$$\rho_{1,3}^y = \frac{\sigma_y^2(t_1 - t_0) + a^2 k_1 k_3 \sigma_z^2}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2} \sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}}, \text{ and}$$

$$\rho_{2,3}^y = \frac{\sigma_y^2(t_2 - t_0) + a^2 k_2 k_3 \sigma_z^2}{\sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2} \sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}},$$

$$\mu_y = a \begin{bmatrix} a_1 \left(\mu_1 - \frac{1}{2} \sigma_1^2 - \sigma_1 \theta_w (1 + \rho_{2,1} + \rho_{3,1} + \rho_{4,1} + \rho_{5,1}) \right) \\ + a_2 \left(\mu_2 - \frac{1}{2} \sigma_2^2 - \sigma_2 \theta_w (1 + \rho_{1,2} + \rho_{3,2} + \rho_{4,2} + \rho_{5,2}) \right) + \dots \\ + a_5 \left(\mu_5 - \frac{1}{2} \sigma_5^2 - \sigma_5 \theta_w (1 + \rho_{1,5} + \rho_{2,5} + \rho_{3,5} + \rho_{4,5}) \right) \end{bmatrix}, \text{ and}$$

$$\sigma_y = a \sqrt{[a_1 \sigma_1 \ a_2 \sigma_2 \ a_3 \sigma_3 \ a_4 \sigma_4 \ a_5 \sigma_5] \begin{pmatrix} 1 & \dots & \rho_{15} \\ \vdots & \ddots & \vdots \\ \rho_{51} & \dots & 1 \end{pmatrix} [a_1 \sigma_1 \ a_2 \sigma_2 \ a_3 \sigma_3 \ a_4 \sigma_4 \ a_5 \sigma_5]'}.}$$

Substituting Equations (24)-(26) and Proposition 2 into Equation (20), we determine $E^Q[1_{\{S_T < 1\}} | Y_{\max} = Y_{t_j}]$ for $j = 1, 2, 3$. After obtaining $E^Q[1_{\{S_T < 1\}} | Y_{\max} = Y_{t_1}]$, $E^Q[1_{\{S_T < 1\}} | Y_{\max} = Y_{t_2}]$, and $E^Q[1_{\{S_T < 1\}} | Y_{\max} = Y_{t_3}]$, we can complete $E^Q(1_{\{S_T < 1\}})$ through Proposition 2, because

$$\begin{aligned} & E^Q(1_{\{S_T < 1\}}) \\ &= E^Q[1_{\{S_T < 1\}} | Y_{\max} = Y_{t_1}] P_r^Q(Y_{\max} = Y_{t_1}) + \\ & E^Q[1_{\{S_T < 1\}} | Y_{\max} = Y_{t_2}] * P_r^Q(Y_{\max} = Y_{t_2}) + E^Q[1_{\{S_T < 1\}} | Y_{\max} = Y_{t_3}] * P_r^Q(Y_{\max} = Y_{t_3}). \end{aligned}$$

However, $E^Q[S_T 1_{\{S_T < 1\}}]$ still must be derived by means of Equation (6), as

proposed by Denuit et al. (2007). From Equation (6), we know

$$E^Q[S_T 1_{\{S_T < 1\}}] = \int_0^1 [1 - \Phi(\Phi^{-1}(P_r(S_T 1_{\{S_T < 1\}} < p)) + \theta)] dp, \quad (27)$$

in which $P_r(\cdot)$ represents the probability under an original probability measure; θ is a

risk premium; and $0 \leq p \leq 1$. Moreover,

$$\begin{aligned}
& P_r \left(S_T 1_{\{S_T < 1\}} < p \right) \\
&= P_r \left(Y_{\max} = Y_{t_1}, Y_{t_1} < K_1 \right) + P_r \left(Y_{\max} = Y_{t_2}, Y_{t_2} < K_1 \right) + P_r \left(Y_{\max} = Y_{t_3}, Y_{t_3} < K_1 \right) \\
&+ P_r \left(Y_{\max} = Y_{t_1}, Y_{t_1} < (K_1 + p(K_2 - K_1)), K_1 \leq Y_{t_1} \leq K_2 \right) \\
&+ P_r \left(Y_{\max} = Y_{t_2}, Y_{t_2} < (K_1 + p(K_2 - K_1)), K_1 \leq Y_{t_2} \leq K_2 \right) \\
&+ P_r \left(Y_{\max} = Y_{t_3}, Y_{t_3} < (K_1 + p(K_2 - K_1)), K_1 \leq Y_{t_3} \leq K_2 \right). \tag{28}
\end{aligned}$$

Following the same procedure in Appendix C, $P_r(Y_{\max} = Y_{t_j}, Y_{t_j} < V)$ for

$j = 1, 2, 3$ are illustrated as Equation (29)-(31) with $V = K_1$ or $K_1 + p(K_2 - K_1)$.

$$\begin{aligned}
& P_r(Y_{\max} = Y_{t_1}, Y_{t_1} < V) = \\
& \left\{ \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \Phi \left(\frac{\ln V - \mu_y^*(t_1 - t_0) - \ln Y_{t_0} - a k_1 u_z}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2}} \right) \right. \\
& \left. - \left(1 - \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right) \right\} \\
& \times \left[1 - \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_2) - a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right] \\
& + \left\{ \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \Phi \left(\frac{\ln K_1 - \mu_y^*(t_1 - t_0) - \ln Y_{t_0} - a k_1 u_z}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2}} \right) \right. \\
& \left. - \left(1 - \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \Phi \left(\frac{-\mu_y^*(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right) \right\} \\
& \times \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_2) - a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right], \tag{29}
\end{aligned}$$

$$\begin{aligned}
P_r(Y_{max} = Y_{t_2}, Y_{t_2} < V) = & \\
& \left\{ \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\ln V - \mu_y^*(t_2 - t_0) - \ln Y_{t_0} - a k_2 u_z}{\sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2}} \right) \right. \\
& \left. - \left(1 - \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_2) - a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right) \right\} \\
& \times \left[1 - \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right] \\
& \left\{ \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\ln V - \mu_y^*(t_2 - t_0) - \ln Y_{t_0} - a k_2 u_z}{\sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2}} \right) \right. \\
& \left. + \left(1 - \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y^*(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right) \right\} \\
& \times \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right], \quad (30)
\end{aligned}$$

and

$$\begin{aligned}
P_r(Y_{max} = Y_{t_3}, Y_{t_3} < V) = & \left\{ \sum_{k_3=0}^{\infty} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\ln V - \mu_y^*(t_3 - t_0) - \ln Y_{t_0} - a k_3 u_z}{\sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}} \right) \right. \\
& \left. - \left(1 - \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right) \right\} \\
& \times \left[1 - \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y^*(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right] \\
& + \left\{ \sum_{k_3=0}^{\infty} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\ln V - \mu_y^*(t_3 - t_0) - \ln Y_{t_0} - a k_3 u_z}{\sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}} \right) \right. \\
& \left. - \left(1 - \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_2) - a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right) \right\} \\
& \times \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y^*(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right], \quad (31)
\end{aligned}$$

for $V = K_1$ or $K_1 + p(K_2 - K_1)$, and $\mu_y^* = a \sum_{i=1}^5 a_i (\mu_i - \frac{1}{2} \sigma_i^2)$.

Appendix D is demonstrated the proofs of Equations (29)-(31) in detail.

Further, we also can derive $P_r(S_T 1_{\{S_T < 1\}} < p)$ and $E^Q[S_T 1_{\{S_T < 1\}}]$. Finally, we thus determine the fair spread rate of the mortality-linked bond.

4. Empirical and Numerical Results

In this section, we use the weekly death rate data from the short-term mortality fluctuations (STMF) data series, established by the Human Mortality Database (HMD)

team to estimate the parameters for the proposed pandemic mortality model shown in Equation (1). We estimate the parameters of $(\mu_i, \sigma_i, u_{\Lambda_i}, \sigma_{\Lambda_i}, u_{\pi_i}, \sigma_{\pi_i}, \lambda_{T_i}, \lambda_{J_i})$ for the five countries of Unites States, England and Wales, France, Italy, and Switzerland. To ensure that we focus on the COVID-19 event and limit the common time for all five countries, we specify a time window from January 1, 2015, to May 24, 2021. Based on the weekly data, it produces 333 observations across five countries for this period. Furthermore, we decompose this full period into two subperiods: pre-COVID-19 and during COVID-19. With the parameter estimates, we can obtain the fair spread of the mortality-linked bonds when considering the pandemic mortality risk, in accordance with the analytical solution derived in Section 3. We also provide sensitivity analyses.

4.1 Descriptive Statistics and Data Analysis

Table 1 contains the descriptive statistics for mortality rates for the five countries across the three periods: the full period, the pre-COVID-19 period, and during COVID-19. In Table 1, we find that the distributions of mortality rates are significantly non-normal at a 5% significance level by means of Jarque-Bera statistic for the full period, the pre-COVID-19 period, and the during COVID-19 period. The volatilities of mortality rates during the COVID-19 period are obviously higher than those in the pre-COVID-19 period in all five countries. The mean of the mortality rates also are far smaller in the pre-COVID-19 period than those during COVID-19 in the Unites States, England and Wales, France, Italy, and Switzerland. Table 1 confirms that COVID-19 caused a significant increase of mortality rates around the world. Table 2 demonstrates

the correlation coefficient matrix of mortality rates for the Unites States, England and Wales, France, Italy, and Switzerland.

Table 1: Descriptive Statistics of Mortality Rates over Various Periods

Statistics	USA	England and Wales	France	Italy	Switzerland
<i>Panel A: Full Period</i>					
Mean	0.008970	0.009313	0.009321	0.011122	0.008067
SD	0.001069	0.001778	0.001175	0.001778	0.001216
J-B	61.150340	120.195110	90.134623	155.482114	149.648116
P-Value	0	0	0	0	0
Obs.	333	333	333	333	333
<i>Panel B: Pre-COVID-19</i>					
Mean	0.008587	0.009032	0.009090	0.010720	0.007899
SD	0.000538	0.001215	0.000956	0.001295	0.000860
J-B	55.020450	111.750600	81.069890	152.276800	147.406300
P-Value	0	0	0	0	0
Obs.	259	259	259	259	259
<i>Panel C: During COVID-19</i>					
Mean	0.010310	0.010300	0.010130	0.012530	0.008660
SD	0.001350	0.002810	0.001480	0.002420	0.001910
J-B	19.962080	37.468850	7.1187050	14.199500	22.773320
P-Value	0.00013	0	0.028457	0.00083	0.00001
Obs.	74	74	74	74	74

Notes: SD = standard deviation, J-B = Jarque-Bera statistic.

Table 2: Correlation Coefficient Matrix for the Unites States, England and Wales, France, Italy, and Switzerland.

	USA	England and Wales	France	Italy	Switzerland
USA	1	0.7034	0.6842	0.7123	0.7109
England and Wales	0.7034	1	0.5542	0.5973	0.5434
France	0.6842	0.5542	1	0.8886	0.7937
Italy	0.7123	0.5973	0.8886	1	0.8047

Switzerland	0.7109	0.5434	0.7937	0.8047	1

4.2 Parameter Estimation and Goodness of Fit of the Pandemic Mortality Model

In this section, we apply a calibration approach to estimate the parameters $(\mu_i, \sigma_i, u_{\Lambda_i}, \sigma_{\Lambda_i}, u_{\pi_i}, \sigma_{\pi_i}, \lambda_{\Gamma_i}, \lambda_{\Gamma_i})$ of the pandemic mortality model shown in Equation (1) for the five countries². The parameters are obtained by minimizing the nonlinear sum of squared errors (SSE) of the actual and the estimated mortality rate. To express the actual and the estimated mortality rate, we rewrite Equation (1) as

$$q_{i,t} = q_{i,0} e^{(\mu_i - 0.5\sigma_i^2)\Delta t + \sigma_i R \varepsilon_i \sqrt{\Delta t}} \prod_{j=1}^{\Gamma_{i,t}} \Lambda_{i,j} \prod_{j=1}^{\tilde{\Gamma}_i} \pi_{i,j}, \text{ for } i = 1, 2, 3, \dots, m. \quad (32)$$

where ε_i is a random variable distributed in normality with mean of 0 and volatility of 1; $\Delta t = \frac{t}{n}$, in which n denotes time steps; R is a correlated random variable through a Cholesky decomposition. Let $d(\ln q_{i,t})$ be the model log rates from Equation (32) and $\hat{d}(\ln q_{i,t})$ be their estimated log rates. The difference $d(\ln q_{i,t}) - \hat{d}(\ln q_{i,t})$ is a function of the values taken by $\Theta = (\mu_i, \sigma_i, u_{\Lambda_i}, \sigma_{\Lambda_i}, u_{\pi_i}, \sigma_{\pi_i}, \lambda_{\Gamma_i}, \lambda_{\Gamma_i})$. Thus, we obtain the parameter vector Θ by solving SSE, which is

$$\text{SSE} = \min_{\Theta} \sum_{j=1}^n \left| \varepsilon_j[\Theta] \right|^2 \quad (33).$$

We use the correlation coefficient matrix in Table 2 and the initial values of

² Calibration refers to the task of estimating the best fitting parameters in a parametric model in comparison with a chosen observable quantity. The comparative information typically consists of historical data for liquid instruments.

$(\mu_i, \sigma_i, u_{\Lambda_i}, \sigma_{\Lambda_i}, u_{\pi_i}, \sigma_{\pi_i}, \lambda_{\Gamma_i}, \lambda_{I_i})$ in Table 3 to find the parameter vector Θ in Equation (33). We use the mean and volatility of the mortality rates as the initial values for each country in various periods. λ_{I_0} symbolizes the initial value of pandemic jump intensity and we calculate it using the mean of pandemic jump frequency caused by $N_{i,t} \geq N_t$ for each country in various periods. λ_{Γ_i} denotes the initial values of the country jump intensity and we use 5 as the initial value of pandemic jump intensity. Finally, q_{i,t_0} is the initial value of mortality rate and we use the mean of mortality rate for each country in various periods.

Table 3: Initial Values of the Calibrated Parameters for Various Periods

Parameters	USA	England and Wales	France	Italy	Switzerland
<i>Panel A: Full Period</i>					
μ_i	-0.000182	-0.000721	-0.000242	-0.000421	-0.000343
σ_i	0.009262	0.0503023	0.018429	0.021980	0.0229931
u_{π_i}	0.001	0.001	0.001	0.001	0.001
σ_{π_i}	0.01	0.01	0.01	0.01	0.01
u_{Λ_i}	0.001	0.001	0.001	0.001	0.001
σ_{Λ_i}	0.01	0.01	0.01	0.01	0.01
λ_{Γ_i}	5	5	5	5	5
q_{i,t_0}	0.008970	0.009313	0.009321	0.011122	0.008067
λ_{I_0}	12	8	10	8	3
<i>Panel B: Pre-COVID-19</i>					
μ_i	0.008587	0.009032	0.009090	0.010720	0.007899
σ_i	0.000538	0.001216	0.000956	0.001295	0.000860
u_{π_i}	0.001	0.001	0.001	0.001	0.001
σ_{π_i}	0.01	0.01	0.01	0.01	0.01
u_{Λ_i}	0.001	0.001	0.001	0.001	0.001
σ_{Λ_i}	0.01	0.01	0.01	0.01	0.01
λ_{Γ_i}	5	5	5	5	5

q_{i,t_0}	0.008587	0.009032	0.009090	0.010720	0.007899
λ_{i_0}	0.001	0.001	0.001	0.001	0.001
<i>Panel C: During COVID-19</i>					
μ_i	0.010314	0.010298	0.010134	0.012533	0.008656
σ_i	0.001353	0.002813	0.001481	0.002422	0.001914
u_{π_i}	0.001	0.001	0.001	0.001	0.001
σ_{π_i}	0.01	0.01	0.01	0.01	0.01
u_{Λ_i}	0.001	0.001	0.001	0.001	0.001
σ_{Λ_i}	0.01	0.01	0.01	0.01	0.01
λ_{Γ_i}	5	5	5	5	5
q_{i,t_0}	0.010314	0.010298	0.010134	0.012533	0.008656
λ_{i_0}	12	8	10	8	3

Notes: The initial values of μ_i and σ_i are the means and volatilities of the mortality rates for the United States, England and Wales, France, Italy, and Switzerland in various periods. Then λ_{i_0} symbolizes the initial values of pandemic jump intensities, which is the means of pandemic jump frequencies caused by $N_{i,t} \geq N_t$ for these countries in various periods. Finally, q_{i,t_0} is the initial values of mortality rates, which are the means of mortality indices for all five countries in various periods.

Then in Figures 3 to 7, we depict the estimated values of the logarithmic mortality rates in the pandemic mortality model; the mean squared errors are 0.0001046, 0.0000108, 0.0002581, 0.0003141, and 0.0003165 for the United States, England and Wales, France, Italy, and Switzerland in the full period, respectively. This evidence affirms that our pandemic mortality model captures the jump in the actual change rates of mortality rates in different countries. The estimated parameters for the proposed pandemic mortality model are presented in Table 4. We can observe that λ_{Γ_i} is higher during COVID-19 period than in the pre-COVID-19 period; that is, COVID-19 increases pandemic jump frequency, as we would expect.

Table 4: Parameter Estimates in Dynamic Processes of Mortality Rates through Calibration with the Pandemic Mortality Model

Parameters	USA	England and Wales	France	Italy	Switzerland
<i>Panel A: Full Period</i>					
μ_i	0.0081	0.0115	0.0086	0.0118	0.0095
σ_i	0.0008	0.0033	0.0017	0.0018	0.0017
u_{π_i}	-0.0009	0.0051	0.0011	0.0011	-0.0004
σ_{π_i}	0.0115	0.0127	0.0108	0.0101	0.0097
u_{Λ_i}	0.0014	-0.0042	0.0004	0.0014	0.0014
σ_{Λ_i}	0.0131	0.0108	0.0111	0.0115	0.0118
λ_{Γ_i}	5.0004	4.9980	5.0001	5.0005	5.0003
$\lambda_{\Gamma_i'}$	12.0578	8.1542	10.3576	8.6051	3.0096
<i>Panel B: Pre-COVID-19</i>					
μ_i	0.0098	0.0096	0.0091	0.0112	0.0099
σ_i	0.0011	0.0018	0.0013	0.0018	0.0012
u_{π_i}	0.0001	0.0006	0.0008	0.0009	0.002
σ_{π_i}	0.0114	0.0108	0.0099	0.0101	0.0123
u_{Λ_i}	0.0036	0.0023	0.0016	0.0016	0.0004
σ_{Λ_i}	0.0114	0.0121	0.0102	0.0108	0.0109
λ_{Γ_i}	4.9999	5.0002	4.9999	5.0003	5.0005
$\lambda_{\Gamma_i'}$	0.0095	0.0012	0.0011	0.0013	0.0009
<i>Panel C: During COVID-19</i>					
μ_i	0.0114	0.0123	0.0104	0.0215	0.0028
σ_i	0.0189	0.0026	0.0021	0.0029	0.0075
u_{π_i}	0.0318	0.0036	0.0031	0.0041	0.0038
σ_{π_i}	0.0138	0.0122	0.0156	0.0122	0.0212
u_{Λ_i}	0.0037	0.0008	0.0018	-0.0051	0.0032
σ_{Λ_i}	0.0192	0.0149	0.0138	0.0196	0.0196
λ_{Γ_i}	5.0084	4.9999	5.0008	5.0013	5.0002
$\lambda_{\Gamma_i'}$	13.0154	9.1208	11.6213	9.1511	3.5891

Notes: The parameter estimates come from Equation (1); i = United States, England and Wales, France, Italy, or Switzerland.

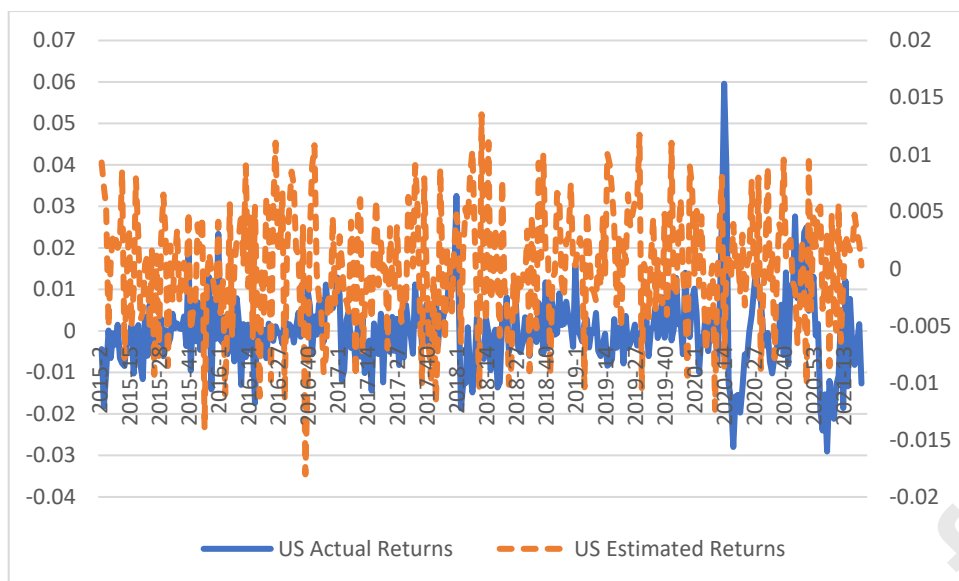


Figure 3. Actual and Estimated Values in the United States

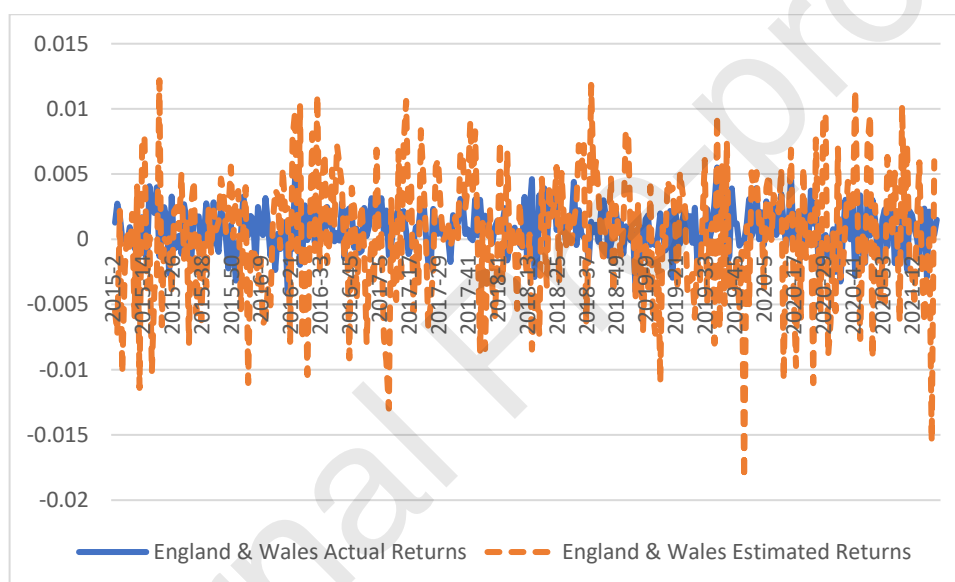


Figure 4. Actual and Estimated Values in England and Wales

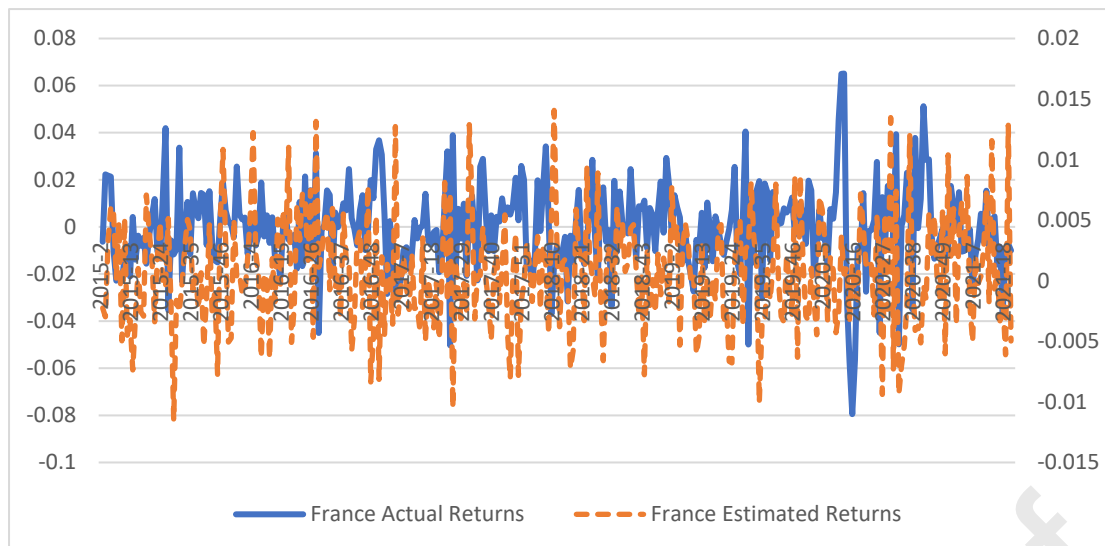


Figure 5. Actual and Estimated Values in France

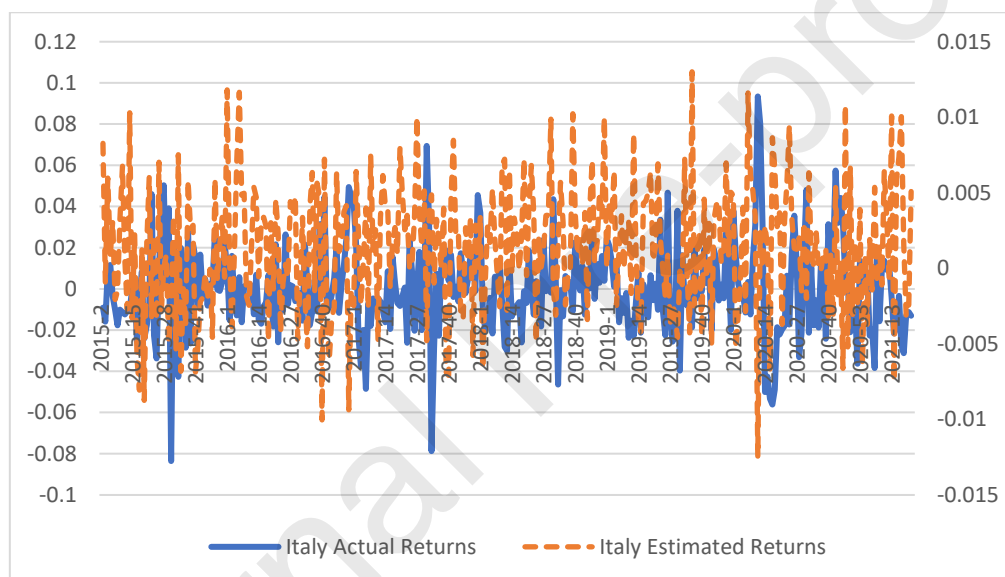


Figure 6. Actual and Estimated Values in Italy

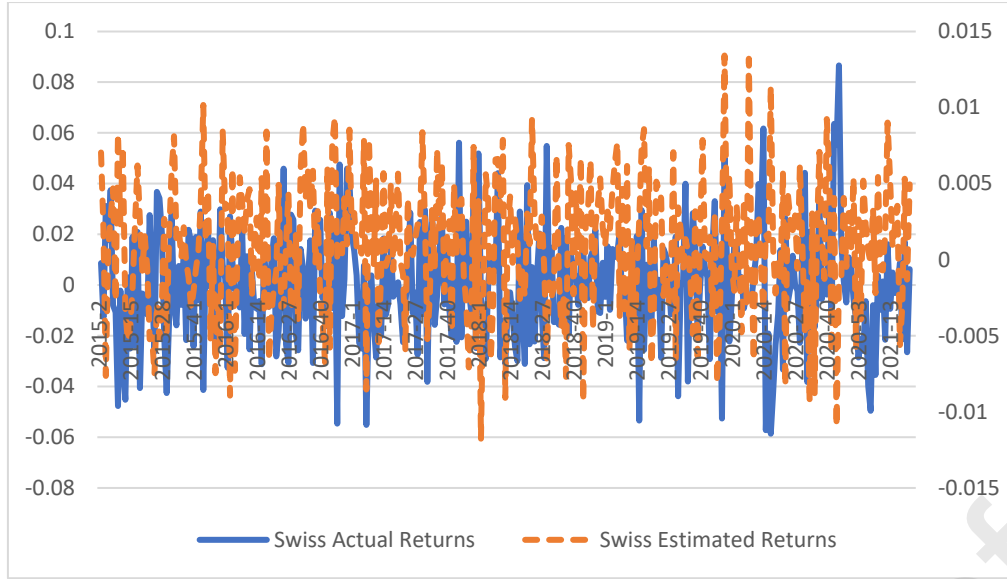


Figure 7. Actual and Estimated Values in Switzerland

Alternatively, Table 5 presents the results of an estimation of the model parameters without pandemic mortality risk. Using the results from Tables 4 and 5; the risk premiums of 0.83, 0.8657, 1.5, and 1.21 offered by, respectively, Cox et al. (2006), Lin and Cox (2008), Chen and Cox (2009), and Lin et al. (2013); and values of $a_1 = 0.7$, $a_2 = 0.15$, $a_3 = 0.075$, $a_4 = 0.05$, and $a_5 = 0.025$, we can calculate the fair spreads of Swiss Re bonds with and without pandemic mortality risk for the five countries. As the results shown in Table 6, the phenomena are consistent. The fair spreads with pandemic mortality risk are far greater than those without pandemic mortality risk across periods; the fair spreads when we include pandemic mortality risk are the highest in all countries during the COVID-19 period. The fair spreads of Swiss Re bonds in the pandemic mortality model are far greater than the 0.45% indicated by Cox et al. (2006). That is, ignoring the effects of pandemic mortality risk leads to significant underestimates of the par spreads for mortality-linked bonds, whereas accounting for this phenomenon reveals the true impacts of pandemic mortality risk in the real world, particularly during the pandemic period.

Table 5: Estimation of Model Parameters without Pandemic Mortality Risk

Parameters	USA	England and Wales	France	Italy	Switzerland
<i>Panel A: Full Period</i>					
μ_i	0.0090	0.0091	0.0079	0.0010	0.0080
σ_i	0.0014	0.0017	0.0011	0.0010	0.0013
u_{Λ_i}	0.0007	0.0009	0.0023	0.0002	0.0007
σ_{Λ_i}	0.0111	0.0104	0.0098	0.0090	0.0101
λ_{Γ_i}	5.0001	5.0001	4.9998	5.0005	5.0001
<i>Panel B: Pre-COVID-19</i>					
μ_i	0.0092	0.0081	0.01	0.0121	0.0099
σ_i	0.0029	0.0004	0.0028	0.0019	0.0011
u_{Λ_i}	0.0017	0.0009	0.0013	0.0012	-0.0009
σ_{Λ_i}	0.0114	0.0105	0.01	0.0109	0.0097
λ_{Γ_i}	5.0076	5.0002	5.0003	5.0004	5.0002
<i>Panel C: During COVID-19</i>					
μ_i	0.0096	0.0087	0.0095	0.011	0.0094
σ_i	0.0006	0.0026	0.0021	0.0011	0.0055
u_{Λ_i}	-0.004	0.0019	0.001	0.0013	0.0036
σ_{Λ_i}	0.0113	0.0132	0.0098	0.0102	0.0158
λ_{Γ_i}	4.9999	4.9986	5.0001	4.9998	5.0016

Table 6: Fair Spreads of Swiss Re Bonds with Various Risk Premiums

Model	$\theta=0.83$	$\theta=0.8657$	$\theta=1.21$	$\theta=1.5$
<i>Panel A: Full Period</i>				
Pandemic Mortality Model	0.2514	0.2692	0.2701	0.2702
No Pandemic Mortality Model	0.0305	0.0322	0.0411	0.0437
<i>Panel B: Pre-COVID-19</i>				
Pandemic Mortality Model	0.0918	0.0989	0.0992	0.0988
No Pandemic Mortality Model	0.0305	0.0308	0.0397	0.0387
<i>Panel C: During COVID-19</i>				
Pandemic Mortality Model	0.2701	0.2711	0.2721	0.2722
No Pandemic Mortality Model	0.0341	0.0340	0.0513	0.0500

4.3 Numerical and Sensitivity Analyses

Using the information from Table 4 and values of $a_1 = 0.7$, $a_2 = 0.15$, $a_3 = 0.075$, $a_4 = 0.05$, $a_5 = 0.025$, $\theta_w = \theta = 0.83$, and face value = 1, we also investigate the price of the Swiss Re bonds under the proposed pandemic mortality model numerically. Table 7 contains several consistent results. First, when the model parameters $(\mu_i, \sigma_i, u_{\Lambda_i}, \sigma_{\Lambda_i}, u_{\pi_i}, \sigma_{\pi_i}, \lambda_{\Gamma_i}, \lambda_{\Gamma_i})$ increase by 10%, the fair spreads are higher than when these model parameters fall by 10% in each country, consistently for the various periods. Second, the relationship between the fair spreads of the bond and model parameters of $(\mu_i, \sigma_i, u_{\Lambda_i}, \sigma_{\Lambda_i}, u_{\pi_i}, \sigma_{\pi_i}, \lambda_{\Gamma_i}, \lambda_{\Gamma_i})$ is positive. The fair spreads thus appear to increase to compensate for pandemic mortality risk as mortality rates increase.

Third, when we compare the sensitivities of the model parameters in the pre-COVID-19 and during COVID-19 periods, we find that they are mostly significantly higher during COVID-19. This evidence confirms the reality: COVID-19 has given rise to massive increases in deaths around the world, and the spreads of mortality-linked bonds increase to compensate for the related pandemic mortality risk. The sensitivity of the volatility of the mortality rates is smallest when the model parameters decline by 10% in the pre-COVID-19 period, but the sensitivity of the volatility of pandemic jump sizes is greatest during the COVID-19 period when σ_{π_i} increases by 10%. Therefore, pandemic mortality risk is a crucial factor for mortality-linked bond pricing during COVID-19. Fourth, the sensitivities generally are greater when each model parameter rises by 10% compared with when the same model parameters decrease by 10% during the COVID-19 period, especially in relation to u_{π_i} , σ_{π_i} , and λ_{Γ_i} . In summary, we show that it is critical to consider pandemic mortality risk when a pandemic event occurs.

Table 7: Fair Spreads of Swiss Re Bond by Face Value in Pandemic Mortality Model for Different Model Parameters

Baseline: Input from Table 4; $a_1 = 0.7$, $a_2 = 0.15$, $a_3 = 0.075$, $a_4 = 0.05$, $a_5 = 0.025$; $\theta_w = \theta = 0.83$; face value = 1.

Model Parameters	Full Period ($C^* = 0.2514$)	Pre-COVID-19 Period ($C^* = 0.0918$)	During COVID-19 Period ($C^* = 0.2701$)
Mean of mortality indices $\mu_i \uparrow 10\%$, for $i = 1, 2, 3, 4, 5$ ⁽⁴⁾ $\mu_i \downarrow 10\%$, for $i = 1, 2, 3, 4, 5$.	0.2771(+1.0222) ⁽¹⁾ 0.2149(+1.4519)	0.1018 (+1.0893) ⁽²⁾ 0.0892 (+0.2832)	0.2999(+1.033) ⁽³⁾ 0.2586(+0.4258)
Volatility of mortality indices $\sigma_i \uparrow 10\%$, for $i = 1, 2, 3, 4, 5$. $\sigma_i \downarrow 10\%$, for $i = 1, 2, 3, 4, 5$.	0.3107(+2.3588) 0.1651(+0.3433)	0.0931(+0.1416) 0.0881(+0.4031)	0.2841(+0.5183) 0.2562(+0.5146)
Mean of specific jump magnitude $u_{\Lambda_i} \uparrow 10\%$, for $i = 1, 2, 3, 4, 5$. $u_{\Lambda_i} \downarrow 10\%$, for $i = 1, 2, 3, 4, 5$.	0.2561(+0.1870) 0.1328(+4.7176)	0.0931(+0.1416) 0.0843(+0.8170)	0.2824(+0.4554) 0.2015(+2.5398)
Volatility of specific jump magnitude $\sigma_{\Lambda_i} \uparrow 10\%$, for $i = 1, 2, 3, 4, 5$. $\sigma_{\Lambda_i} \downarrow 10\%$, for $i = 1, 2, 3, 4, 5$.	0.2617(+0.4097) 0.1321(+4.7454)	0.0977(+0.6427) 0.0898(+0.2179)	0.2907(+0.7627) 0.2502(+0.7368)
Mean of pandemic jump magnitude $u_{\pi_i} \uparrow 10\%$, for $i = 1, 2, 3, 4, 5$. $u_{\pi_i} \downarrow 10\%$, for $i = 1, 2, 3, 4, 5$.	0.2918(+1.6070) 0.2412(+0.4057)	0.0937(+0.2070) 0.0891(+0.2941)	0.2932(+0.8552) 0.2551(+0.5553)
Volatility of pandemic jump magnitude $\sigma_{\pi_i} \uparrow 10\%$, for $i = 1, 2, 3, 4, 5$. $\sigma_{\pi_i} \downarrow 10\%$, for $i = 1, 2, 3, 4, 5$.	0.2628(+0.4535) 0.2451(+0.2506)	0.1011(+1.0131) 0.0838(+0.8715)	0.3952(+4.6316) 0.2345(+1.3180)

Pandemic jump intensity $\lambda_{i^*} \uparrow 10\%$, for $i = 1, 2, 3, 4, 5$. $\lambda_{i^*} \downarrow 10\%$, for $i = 1, 2, 3, 4, 5$.	0.2561(+0.1870) 0.2481(+0.1313)	0.0925(+0.0763) 0.0856(+0.6754)	0.2965(+0.9774) 0.2458(+0.8997)
Specific jump intensity $\lambda_{T_{i^*}} \uparrow 10\%$, for $i = 1, 2, 3, 4, 5$. $\lambda_{T_{i^*}} \downarrow 10\%$, for $i = 1, 2, 3, 4, 5$.	0.2671(+0.6245) 0.2245(+1.0700)	0.0984(+0.7190) 0.0842(+0.8279)	0.2768(+0.9184) 0.2317(+1.4217)

Notes: (1) 1.0222 is calculated as $\frac{(0.2771 - 0.2514) / 0.2514}{+0.1}$, which is the sensitivity of mean of mortality indices in the full period;

(2) 1.0893 is determined as $\frac{(0.1018 - 0.0918) / 0.0918}{+0.1}$, which is the sensitivity of mean of mortality indices during the pre-COVID-19 period;

(3) 1.1033 is determined as $\frac{(0.2999 - 0.2701) / 0.2701}{+0.1}$, which is the sensitivity of mean of mortality indices during the COVID-19 period;

(4) $i = 1, 2, 3, 4, 5$ represent the United States, England and Wales, France, Italy, and Switzerland, respectively;

(5) Sensitivities of model parameters are in parentheses; and

(6) C^* stands for the fair spreads of Swiss Re bond in various periods.

5. Conclusion

Transferring mortality losses using mortality-linked securities is critical to the insurance industry. Many life insurers operate their businesses internationally. Using patterns of mortality experience, we find that a pandemic event can cause significant pandemic jump of deaths across countries respectively. Although prior studies have considered mortality rates with jumps, they explain the co-movement of mortality rates using common jumps across countries. Instead, mortality trends offer empirical evidence that pandemic mortality jump may occur in a country when a pandemic event results in globally massive deaths and the new death numbers in the country are greater or equal to the average new death in the world. Existing research rarely has modeled this phenomenon of pandemic mortality rates considering the pandemic jump. To fill that gap, the current article offers a fresh look at the effects of pandemic mortality risk on the valuation of mortality securities. Employing a threshold jump, we develop a pandemic mortality model. Using the Wang (2000) transform, Cox et al. (2006) and Denuit et al.'s (2007) approach, we derive a valuation formula for Swiss Re bonds, based on our pandemic mortality model.

The empirical analysis reveals that the fair par spreads of the mortality fixed-coupon bond in our model are far higher than those obtained using Cox et al.'s (2006) method, which are significantly greater during the COVID-19 period than in the pre-COVID-19 period. Accounting for pandemic mortality risk enables the fair spread of mortality bonds to match the real world better, especially during COVID-19, which is beneficial to efforts to price mortality-linked securities and manage pandemic mortality risk for reinsurers in the post-pandemic world.

In our analyses in this article, we assume the population of newborns is static and ignore dynamic population effects, such as due to immigration. However, dynamic populations represent realistic issues, which are worth exploring in further research.

Appendix A

From Cox et al. (2006), we obtain the following relationship between $W_{i,t}$ and $W_{i,t}^Q$:

$$W_{i,t}^Q = W_{i,t} + \beta_{w_i} t, \quad (\text{A.1})$$

where $\beta_{w_i} = \theta_{w_i} + \sum_{j=1}^5 \rho_{i,j} \theta_{w_j}$ for $i \neq v$, i or $v = 1, 2, \dots, 5$. Following Cox et al. (2006),

we assume θ_{w_i} and θ_{w_j} are equal, which means $\theta_{w_i} = \theta_{w_j} = \theta_w$. Thus, $\beta_{w_i} = \theta_w (1 + \sum_{j=1}^5 \rho_{i,j})$

for $i \neq v$, i or $v = 1, 2, \dots, 5$. Substituting Equation (A.1) into Equation (12) produces

$$\ln q_{i,T} = \ln q_{i,t_0} + (\mu_i - \frac{1}{2} \sigma_i^2 - \sigma_i \beta_{w_i})(T - t_0) + \sigma_i W_{i,T-t_0}^Q + \sum_{l=1}^{X_{i,T}} \ln x_{i,l}. \quad (\text{A.2})$$

From Equation (A.2) and $Y_{t_i} = (q_{1,t_i}^{a_1} q_{2,t_i}^{a_2} \dots q_{5,t_i}^{a_5})^{\frac{1}{a_1+a_2+\dots+a_5}}$, we know

$$\ln Y_T = \ln Y_{t_0} + \mu_y (T - t_0) + \sigma_y W_{T-t_0}^Q + a \sum_{l=1}^{X_T} \ln Z_l, \quad (\text{A.3})$$

where $a = \frac{1}{a_1 + a_2 + \dots + a_5}$;

$$\mu_y = a \left[\begin{aligned} & a_1 \left(\mu_1 - \frac{1}{2} \sigma_1^2 - \sigma_1 \theta_w (1 + \rho_{2,1} + \rho_{3,1} + \rho_{4,1} + \rho_{5,1}) \right) \\ & + a_2 \left(\mu_2 - \frac{1}{2} \sigma_2^2 - \sigma_2 \theta_w (1 + \rho_{1,2} + \rho_{3,2} + \rho_{4,2} + \rho_{5,2}) \right) + \dots \\ & + a_5 \left(\mu_5 - \frac{1}{2} \sigma_5^2 - \sigma_5 \theta_w (1 + \rho_{1,5} + \rho_{2,5} + \rho_{3,5} + \rho_{4,5}) \right) \end{aligned} \right];$$

$$\sigma_y = a \sqrt{[a_1 \sigma_1 \quad a_2 \sigma_2 \quad a_3 \sigma_3 \quad a_4 \sigma_4 \quad a_5 \sigma_5] \begin{bmatrix} 1 & \dots & \rho_{15} \\ \vdots & \ddots & \vdots \\ \rho_{51} & \dots & 1 \end{bmatrix} [a_1 \sigma_1 \quad a_2 \sigma_2 \quad a_3 \sigma_3 \quad a_4 \sigma_4 \quad a_5 \sigma_5]};$$

$$\text{and } \sum_{l=1}^{X_r} \ln Z_l = a_1 \sum_{l=1}^{X_{1,T}} \ln x_{1,l} + a_2 \sum_{l=1}^{X_{2,T}} \ln x_{2,l} + \dots + a_5 \sum_{l=1}^{X_{5,T}} \ln x_{5,l}, \quad \ln Z \sim N(u_z, \sigma_z^2).$$

Thus, Proposition 1 is confirmed.

Appendix B

Because

$$\begin{aligned} P_r^Q(Y_{\max} = Y_{t_1}) &= P_r^Q(Y_{t_1} > Y_{t_2} > Y_{t_3}) + P_r^Q(Y_{t_1} > Y_{t_3} > Y_{t_2}) \\ &= P_r^Q(Y_{t_1} > Y_{t_2}, Y_{t_2} > Y_{t_3}) + P_r^Q(Y_{t_1} > Y_{t_3}, Y_{t_3} > Y_{t_2}), \end{aligned}$$

$(Y_{t_1} - Y_{t_2}) \perp (Y_{t_2} - Y_{t_3})$, and $(Y_{t_1} - Y_{t_3}) \perp (Y_{t_3} - Y_{t_2})$, $P_r^Q(Y_{\max} = Y_{t_1})$ can be rewritten as:

$$P_r^Q(Y_{\max} = Y_{t_1}) = P_r^Q(Y_{t_1} > Y_{t_2}) P_r^Q(Y_{t_2} > Y_{t_3}) + P_r^Q(Y_{t_1} > Y_{t_3}) P_r^Q(Y_{t_3} > Y_{t_2}). \quad (\text{B.1})$$

Due to the independent relation between X_{t_1} and X_{t_3} , $P_r^Q(Y_{t_1} > Y_{t_3})$ also can be written

as:

$$P_r^Q(Y_{t_1} > Y_{t_3}) = P_r^Q(\ln Y_{t_1} > \ln Y_{t_3} | X_{t_1} = k_1, X_{t_3} = k_3) P_r^Q(X_{t_1} = k_1) P_r^Q(X_{t_3} = k_3). \quad (\text{B.2})$$

By applying Proposition 1, we use Equation (B.2) to derive

$$P_r^Q(Y_{t_1} > Y_{t_3}) = \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right). \quad (\text{B.3})$$

With the same procedure, $P_r^Q(Y_{t_1} > Y_{t_2})$ and $P_r^Q(Y_{t_2} > Y_{t_3})$ take on the following forms:

$$P_r^Q(Y_{t_1} > Y_{t_2}) = \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right), \quad (\text{B.4})$$

and

$$P_r^Q(Y_{t_2} > Y_{t_3}) = \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi\left(\frac{-\mu_y(t_3-t_2) - a(k_3-k_2)u_z}{\sqrt{\sigma_y^2(t_3-t_2) + a^2(k_3-k_2)\sigma_z^2}}\right). \quad (\text{B.5})$$

Next, by inserting Equations (B.3)–(B.5) into Equation (B.1), we obtain $P_r^Q(Y_{\max} = Y_{t_1})$,

as shown in Equation (B.6). Similarly, $P_r^Q(Y_{\max} = Y_{t_2})$ and $P_r^Q(Y_{\max} = Y_{t_3})$ can be

obtained, as in Equations (B.7) and (B.8), respectively. Thus, we prove Proposition 2.

$$\begin{aligned} P_r^Q(Y_{\max} = Y_{t_1}) &= \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi\left(\frac{-\mu_y(t_2-t_1) - a(k_2-k_1)u_z}{\sqrt{\sigma_y^2(t_2-t_1) + a^2(k_2-k_1)\sigma_z^2}}\right) \right] \\ &\times \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi\left(\frac{-\mu_y(t_3-t_2) - a(k_3-k_2)u_z}{\sqrt{\sigma_y^2(t_3-t_2) + a^2(k_3-k_2)\sigma_z^2}}\right) \right] \quad (\text{B.6}) \\ &+ \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi\left(\frac{-\mu_y(t_3-t_1) - a(k_3-k_1)u_z}{\sqrt{\sigma_y^2(t_3-t_1) + a^2(k_3-k_1)\sigma_z^2}}\right) \right] \\ &\times \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi\left(\frac{\mu_y(t_3-t_2) + a(k_3-k_2)u_z}{\sqrt{\sigma_y^2(t_3-t_2) + a^2(k_3-k_2)\sigma_z^2}}\right) \right], \end{aligned}$$

$$\begin{aligned} P_r^Q(Y_{\max} = Y_{t_2}) &= \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \Phi\left(\frac{\mu_y(t_2-t_1) + a(k_2-k_1)u_z}{\sqrt{\sigma_y^2(t_2-t_1) + a^2(k_2-k_1)\sigma_z^2}}\right) \right] \\ &\times \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi\left(\frac{-\mu_y(t_3-t_1) - a(k_3-k_1)u_z}{\sqrt{\sigma_y^2(t_3-t_1) + a^2(k_3-k_1)\sigma_z^2}}\right) \right] \\ &+ \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi\left(\frac{-\mu_y(t_3-t_2) - a(k_3-k_2)u_z}{\sqrt{\sigma_y^2(t_3-t_2) + a^2(k_3-k_2)\sigma_z^2}}\right) \right] \\ &\times \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi\left(\frac{\mu_y(t_3-t_1) + a(k_3-k_1)u_z}{\sqrt{\sigma_y^2(t_3-t_1) + a^2(k_3-k_1)\sigma_z^2}}\right) \right], \quad (\text{B.7}) \end{aligned}$$

and

$$\begin{aligned}
P_r^Q(Y_{max} = Y_{t_3}) &= \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\mu_y(t_3 - t_2) + a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right] \\
&\times \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\mu_y(t_2 - t_1) + a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right] \\
&+ \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\mu_y(t_3 - t_1) + a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right] \\
&\times \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right]. \quad (B.8)
\end{aligned}$$

Appendix C

Applying Proposition 1 and Appendix B, Equations (C.1) and (C.2) can be derived

as follows.

$$\begin{aligned}
P_r^Q(Y_{t_1} > Y_{t_3}, Y_{t_1} > K_2) &= P_r^Q(Y_{t_1} > Y_{t_3}) - P_r^Q(Y_{t_3} < K_2) + P_r^Q(Y_{t_3} < K_2, Y_{t_1} > K_2) \\
&= \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \\
&- \sum_{k_3=0}^{\infty} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\ln K_2 - \mu_y(t_3 - t_0) - \ln Y_{t_0} - a k_3 u_z}{\sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}} \right) + \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi(-d_1, d_3, \rho_{1,3}^y), \quad (C.1) \text{ and}
\end{aligned}$$

$$\begin{aligned}
P_r^Q(Y_{t_1} > Y_{t_2}, Y_{t_1} > K_2) &= \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \\
&- \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\ln K_2 - \mu_y(t_2 - t_0) - \ln Y_{t_0} - a k_2 u_z}{\sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2}} \right) + \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi(-d_1, d_2, \rho_{1,2}^y). \quad (C.2)
\end{aligned}$$

Therefore, $P_r^Q(Y_{max} = Y_{t_1}, Y_{t_1} > K_2)$ can be derived as illustrated in Equation (C.3).

$$\begin{aligned}
P_r^Q(Y_{max} = Y_{t_1}, Y_{t_1} > K_2) = & \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3-t_1) - a(k_3-k_1)u_z}{\sqrt{\sigma_y^2(t_3-t_1) + a^2(k_3-k_1)\sigma_z^2}} \right) \right. \\
& \left. - \sum_{k_3=0}^{\infty} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\ln K_2 - \mu_y(t_3-t_0) - \ln Y_{t_0} - a k_3 u_z}{\sqrt{\sigma_y^2(t_3-t_0) + a^2 k_3 \sigma_z^2}} \right) \right. \\
& \left. + \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi(-d_1, d_3, \rho_{1,3}^y) \right] \\
& \times \left[1 - \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3-t_2) - a(k_3-k_2)u_z}{\sqrt{\sigma_y^2(t_3-t_2) + a^2(k_3-k_2)\sigma_z^2}} \right) \right] \\
& \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y(t_2-t_1) - a(k_2-k_1)u_z}{\sqrt{\sigma_y^2(t_2-t_1) + a^2(k_2-k_1)\sigma_z^2}} \right) \right. \\
& \left. - \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\ln K_2 - \mu_y(t_2-t_0) - \ln Y_{t_0} - a k_2 u_z}{\sqrt{\sigma_y^2(t_2-t_0) + a^2 k_2 \sigma_z^2}} \right) \right. \\
& \left. + \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi(-d_1, d_2, \rho_{1,2}^y) \right] \\
& \times \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3-t_2) - a(k_3-k_2)u_z}{\sqrt{\sigma_y^2(t_3-t_2) + a^2(k_3-k_2)\sigma_z^2}} \right) \right]. \quad (C.3)
\end{aligned}$$

Similarly, $P_r^Q(Y_{max} = Y_{t_2}, Y_{t_2} > K_2)$ and $P_r^Q(Y_{max} = Y_{t_3}, Y_{t_3} > K_2)$ is shown in

Equations (C.4) and (C.5).

$$\begin{aligned}
P_r^Q(Y_{max} = Y_{t_2}, Y_{t_2} > K_2) = & \\
& \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3 - t_2) - a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right. \\
& \left. - \sum_{k_3=0}^{\infty} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\ln K_2 - \mu_y(t_3 - t_0) - \ln Y_{t_0} - a k_3 u_z}{\sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}} \right) \right. \\
& \left. + \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi(-d_2, d_3, \rho_{2,3}^y) \right] \\
& \times \left[1 - \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right] \\
& \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\mu_y(t_2 - t_1) + a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right. \\
& \left. - \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \Phi \left(\frac{\ln K_2 - \mu_y(t_1 - t_0) - \ln Y_{t_0} - a k_1 u_z}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2}} \right) \right. \\
& \left. + \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi(d_1, -d_2, \rho_{1,2}^y) \right] \\
& \times \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right],
\end{aligned}$$

(C.4) and

$$\begin{aligned}
P_r^Q(Y_{max} = Y_{t_3}, Y_{t_3} > K_2) = & \left\{ \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\mu_y(t_3 - t_2) + a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right. \\
& \left. - \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\ln K_2 - \mu_y(t_2 - t_0) - \ln Y_{t_0} - a k_2 u_z}{\sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2}} \right) \right. \\
& \left. + \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi(d_2, -d_3, \rho_{2,3}^y) \right\} \\
& \times \left[1 - \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right] \\
& \left\{ \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\mu_y(t_3 - t_1) + a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right. \\
& \left. - \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \Phi \left(\frac{\ln K_2 - \mu_y(t_1 - t_0) - \ln Y_{t_0} - a k_1 u_z}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2}} \right) \right\} \\
& + \left\{ \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi(d_1, -d_3, \rho_{1,3}^y) \right\} \\
& \times \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right], \tag{C.5}
\end{aligned}$$

$$\text{with } d_1 = \frac{\ln K_2 - \mu_y(t_1 - t_0) - \ln Y_{t_0} - a k_1 u_z}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2}}, \quad d_2 = \frac{\ln K_2 - \mu_y(t_2 - t_0) - \ln Y_{t_0} - a k_2 u_z}{\sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2}},$$

$$d_3 = \frac{\ln K_2 - \mu_y(t_3 - t_0) - \ln Y_{t_0} - a k_3 u_z}{\sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}}, \quad u_z = \frac{\sum_{i=1}^5 s_i a_i u_{x_i}}{k}, \quad \sigma_z^2 = \frac{\sum_{i=1}^5 s_i a_i^2 \sigma_{x_i}^2}{k},$$

$$\rho_{1,2}^y = \frac{\sigma_y^2(t_1 - t_0) + a^2 k_1 k_2 \sigma_z^2}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2} \sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2}},$$

$$\rho_{1,3}^y = \frac{\sigma_y^2(t_1 - t_0) + a^2 k_1 k_3 \sigma_z^2}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2} \sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}},$$

$$\rho_{2,3}^y = \frac{\sigma_y^2(t_2 - t_0) + a^2 k_2 k_3 \sigma_z^2}{\sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2} \sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}},$$

$$\mu_y = a \begin{bmatrix} a_1 \left(\mu_1 - \frac{1}{2} \sigma_1^2 - \sigma_1 \theta_w (1 + \rho_{2,1} + \rho_{3,1} + \rho_{4,1} + \rho_{5,1}) \right) \\ + a_2 \left(\mu_2 - \frac{1}{2} \sigma_2^2 - \sigma_2 \theta_w (1 + \rho_{1,2} + \rho_{3,2} + \rho_{4,2} + \rho_{5,2}) \right) + \dots \\ + a_5 \left(\mu_5 - \frac{1}{2} \sigma_5^2 - \sigma_5 \theta_w (1 + \rho_{1,5} + \rho_{2,5} + \rho_{3,5} + \rho_{4,5}) \right) \end{bmatrix}, \text{ and}$$

$$\sigma_y = a \sqrt{[a_1 \sigma_1 \ a_2 \sigma_2 \ a_3 \sigma_3 \ a_4 \sigma_4 \ a_5 \sigma_5] \begin{pmatrix} 1 & \dots & \rho_{15} \\ \vdots & \ddots & \vdots \\ \rho_{51} & \dots & 1 \end{pmatrix} [a_1 \sigma_1 \ a_2 \sigma_2 \ a_3 \sigma_3 \ a_4 \sigma_4 \ a_5 \sigma_5]'}.}$$

Appendix D

Because

$$\begin{aligned} & P_r(Y_{\max} = Y_{t_1}, Y_{t_1} < V) \\ &= P_r(Y_{t_1} < V, Y_{t_1} > Y_{t_2}) P_r(Y_{t_2} > Y_{t_3}) + P_r(Y_{t_1} < V, Y_{t_1} > Y_{t_3}) P_r(Y_{t_3} > Y_{t_2}) \\ &= [P_r(Y_{t_1} < V) - P_r(Y_{t_1} < Y_{t_2})] P_r(Y_{t_2} > Y_{t_3}) + [P_r(Y_{t_1} < V) - P_r(Y_{t_1} < Y_{t_3})] P_r(Y_{t_3} > Y_{t_2}), \end{aligned} \quad (\text{D.1})$$

$$\begin{aligned} & P_r(Y_{\max} = Y_{t_2}, Y_{t_2} < V) \\ &= P_r(Y_{t_2} < V, Y_{t_2} > Y_{t_3}) P_r(Y_{t_3} > Y_{t_1}) + P_r(Y_{t_2} < V, Y_{t_2} > Y_{t_1}) P_r(Y_{t_1} > Y_{t_3}) \\ &= [P_r(Y_{t_2} < V) - P_r(Y_{t_2} < Y_{t_3})] P_r(Y_{t_3} > Y_{t_1}) + [P_r(Y_{t_2} < V) - P_r(Y_{t_2} < Y_{t_1})] P_r(Y_{t_1} > Y_{t_3}), \end{aligned} \quad (\text{D.2})$$

$$\begin{aligned} & P_r(Y_{\max} = Y_{t_3}, Y_{t_3} < V) \\ &= P_r(Y_{t_3} < V, Y_{t_3} > Y_{t_1}) P_r(Y_{t_1} > Y_{t_2}) + P_r(Y_{t_3} < V, Y_{t_3} > Y_{t_2}) P_r(Y_{t_2} > Y_{t_1}) \\ &= [P_r(Y_{t_3} < V) - P_r(Y_{t_3} < Y_{t_1})] P_r(Y_{t_1} > Y_{t_2}) + [P_r(Y_{t_3} < V) - P_r(Y_{t_3} < Y_{t_2})] P_r(Y_{t_2} > Y_{t_1}), \end{aligned} \quad (\text{D.3})$$

$$P_r(Y_{t_2} > Y_{t_3}) = \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_2) - a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right), \quad (\text{D.4})$$

$$P_r(Y_{t_2} < Y_{t_1}) = \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y^*(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right), \quad (\text{D.5})$$

$$P_r(Y_{t_3} < Y_{t_1}) = \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right), \quad (\text{D.6}) \text{ and}$$

$$P_r(Y_{t_j} < V) = \sum_{k_j=0}^{\infty} \frac{\lambda_{X_{t_j}}^{k_j} e^{-\lambda_{X_{t_j}}}}{k_j!} \Phi \left(\frac{\ln V - \mu_y^*(t_j - t_0) - \ln Y_{t_0} - a k_j u_z}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_j \sigma_z^2}} \right) \text{ for } j = 1, 2, 3, \quad (\text{D.7})$$

with $\mu_y^* = a \sum_{i=1}^5 a_i (\mu_i - \frac{1}{2} \sigma_i^2)$, and $V = K_1$ or $K_1 + p(K_2 - K_1)$, we can obtain

$P_r(Y_{\max} = Y_{t_1}, Y_{t_1} < V)$, $P_r(Y_{\max} = Y_{t_2}, Y_{t_2} < V)$, and $P_r(Y_{\max} = Y_{t_3}, Y_{t_3} < V)$ as shown

in Equations (D.8)-(D.10).

$$\begin{aligned} P_r(Y_{\max} = Y_{t_1}, Y_{t_1} < V) = & \left\{ \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \Phi \left(\frac{\ln V - \mu_y^*(t_1 - t_0) - \ln Y_{t_0} - a k_1 u_z}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2}} \right) \right. \\ & \left. - \left(1 - \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right) \right\} \\ & \times \left[1 - \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_2) - a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right] \\ & + \left\{ \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \Phi \left(\frac{\ln K_1 - \mu_y^*(t_1 - t_0) - \ln Y_{t_0} - a k_1 u_z}{\sqrt{\sigma_y^2(t_1 - t_0) + a^2 k_1 \sigma_z^2}} \right) \right. \\ & \left. - \left(1 - \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_1} e^{-\lambda_{X_{t_2}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y^*(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right) \right\} \\ & \times \left[\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_2) - a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right], \quad (\text{D.8}) \end{aligned}$$

$$\begin{aligned}
P_r(Y_{max} = Y_{t_2}, Y_{t_2} < V) = & \\
& \left\{ \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\ln V - \mu_y^*(t_2 - t_0) - \ln Y_{t_0} - a k_2 u_z}{\sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2}} \right) \right. \\
& \left. - \left(1 - \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_2) - a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right) \right\} \\
& \times \left[1 - \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right] \\
& \left\{ \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{\ln V - \mu_y^*(t_2 - t_0) - \ln Y_{t_0} - a k_2 u_z}{\sqrt{\sigma_y^2(t_2 - t_0) + a^2 k_2 \sigma_z^2}} \right) \right. \\
& \left. + \left(1 - \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y^*(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right) \right\} \\
& \times \left[\sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right], \quad (D.9) \text{ and}
\end{aligned}$$

$$\begin{aligned}
P_r(Y_{max} = Y_{t_3}, Y_{t_3} < V) = & \left\{ \sum_{k_3=0}^{\infty} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\ln V - \mu_y^*(t_3 - t_0) - \ln Y_{t_0} - a k_3 u_z}{\sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}} \right) \right. \\
& \left. - \left(1 - \sum_{k_3=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_1) - a(k_3 - k_1)u_z}{\sqrt{\sigma_y^2(t_3 - t_1) + a^2(k_3 - k_1)\sigma_z^2}} \right) \right) \right\} \\
& \times \left[1 - \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y^*(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right] \\
& + \left\{ \sum_{k_3=0}^{\infty} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{\ln V - \mu_y^*(t_3 - t_0) - \ln Y_{t_0} - a k_3 u_z}{\sqrt{\sigma_y^2(t_3 - t_0) + a^2 k_3 \sigma_z^2}} \right) \right. \\
& \left. - \left(1 - \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \frac{\lambda_{X_{t_3}}^{k_3} e^{-\lambda_{X_{t_3}}}}{k_3!} \Phi \left(\frac{-\mu_y^*(t_3 - t_2) - a(k_3 - k_2)u_z}{\sqrt{\sigma_y^2(t_3 - t_2) + a^2(k_3 - k_2)\sigma_z^2}} \right) \right) \right\} \\
& \times \left[\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \frac{\lambda_{X_{t_1}}^{k_1} e^{-\lambda_{X_{t_1}}}}{k_1!} \frac{\lambda_{X_{t_2}}^{k_2} e^{-\lambda_{X_{t_2}}}}{k_2!} \Phi \left(\frac{-\mu_y^*(t_2 - t_1) - a(k_2 - k_1)u_z}{\sqrt{\sigma_y^2(t_2 - t_1) + a^2(k_2 - k_1)\sigma_z^2}} \right) \right], \quad (D.10)
\end{aligned}$$

with $\mu_y^* = a \sum_{i=1}^5 a_i (\mu_i - \frac{1}{2} \sigma_i^2)$, and $V = K_1$ or $K_1 + p(K_2 - K_1)$.

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